



2010
TRIAL HSC EXAMINATION

Mathematics Extension 2

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black or blue pen
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this page
- All necessary working should be shown in every question

Total Marks – 120

Attempt Questions 1–8

All questions are of equal value

At the end of the examination, place your solution booklets in order and put this question paper on top. Submit one bundle. The bundle will be separated before marking commences so that anonymity will be maintained.

NAME: _____

TEACHER: _____

NUMBER: _____

QUESTION	MARK
1	/15
2	/15
3	/15
4	/15
5	/15
6	/15
7	/15
8	/15
TOTAL	/120

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Total Marks – 120

Attempt Questions 1–8

All questions are of equal value

Begin each question in a NEW BOOKLET.

Question 1 (15 marks)

Marks

(a) Find $\int \frac{\cos^2 x}{1 - \sin x} dx$. **2**

(b) Use the method of partial fractions to find $\int \frac{1}{x^2 + x} dx$. **3**

(c) (i) Use the table of standard integrals to find **1**

$$\int \frac{dx}{\sqrt{4x^2 - 1}}.$$

(ii) Is the following statement true or false? Justify your answer. **1**

$$\int_{-1}^1 \frac{dx}{\sqrt{4x^2 - 1}} = 2 \int_0^1 \frac{dx}{\sqrt{4x^2 - 1}}.$$

(d) By using the substitution, $t = \tan \frac{\theta}{2}$, evaluate **3**

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{2 + \sin \theta}.$$

(e) (i) Show that $\int_0^1 x^n e^{-x} dx = nI_{n-1} - \frac{1}{e}$ where $I_n = \int_0^1 x^n e^{-x} dx$. **3**

(ii) Hence deduce that $\int_0^1 x^3 e^{-x} dx = 6 - \frac{16}{e}$. **2**

Question 2 (15 marks) Use a SEPARATE writing booklet.

Marks

(a) Given the complex number $z = -7 + 2i$, find

(i) \bar{z} **1**

(ii) $\arg \bar{z}$ giving your answer to one decimal place. **1**

(iii) $\arg iz + \arg \bar{iz}$ **2**

(b) By using the modulus-argument form of a complex number, evaluate **2**

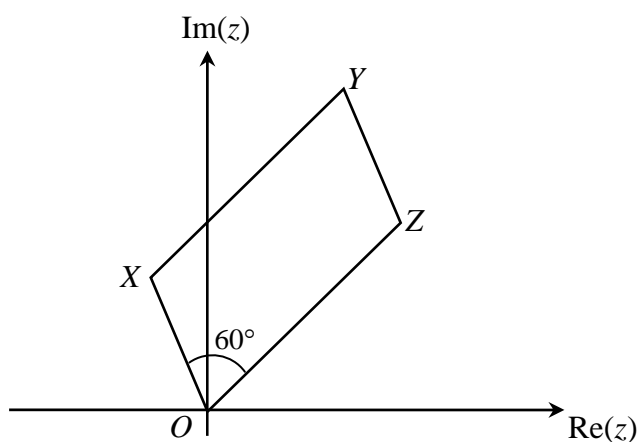
$$(1 - \sqrt{3}i)^9$$

(c) On the Argand diagram, sketch the region described by **3**

$$|z| < 2 \quad \text{and} \quad \frac{2\pi}{3} \leq \arg z \leq \frac{5\pi}{6}$$

(d) In the diagram below $OXYZ$ is a parallelogram with $OX = \frac{1}{2}OZ$

The point X represents the complex number $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$



If $\angle XOZ = 60^\circ$, what complex number does Z represent? **2**

(e) Given that $z = \cos \theta + i \sin \theta$

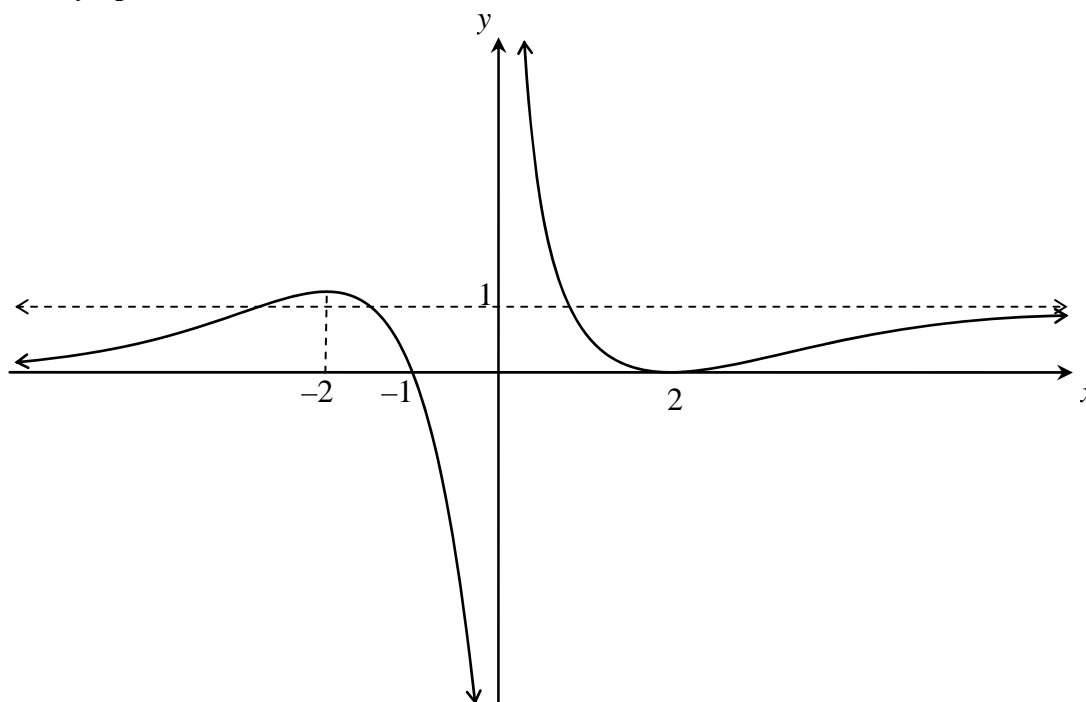
(i) Show that $z^n + \frac{1}{z^n} = 2 \cos n\theta$. **1**

(ii) Hence, or otherwise, solve the equation $2z^4 - z^3 + 3z^2 - z + 2 = 0$. **3**

Question 3 (15 marks) Use a SEPARATE writing booklet.

Marks

- (a) The graph of $y = f(x)$ is displayed below. The lines $y = 1$, $x = 0$ and $y = 0$ are asymptotes.



Using the separate page of graphs provided, sketch each of the graphs below. In each case, clearly label any maxima or minima, intercepts and the equations of any asymptotes.

- (i) $y = f(|x|)$ 2
- (ii) $y^2 = f(x)$ 2
- (iii) $y = e^{f(x)}$ 2
- (iv) $y = \sin^{-1}[f(x)]$ 2
- (b) (i) On the same set of axes, sketch the graphs of $y = e^x$ and $y = \sqrt[3]{x}$. 2
- (ii) On another set of axes sketch, using the graphs in (i), the graph of $y = \sqrt[3]{xe^x}$. 3
- (iii) Use the last sketch to determine the values of k for which the equation $\sqrt[3]{x} = \frac{kx+2}{e^x}$ has exactly one solution. 2

Question 4 (15 marks) Use a SEPARATE writing booklet.

Marks

(a) Consider the equation of the conic below

$$\frac{x^2}{29-\lambda} - \frac{y^2}{4-\lambda} = 1$$

- (i) Find the values of λ for which this conic defines an ellipse. **2**
- (ii) If the equation represents an hyperbola, show that the focus of the hyperbola is independent of λ . **2**
- (iii) Sketch the conic defined by $\lambda = 13$. **3**
- (b) A sequence $\mu_1, \mu_2, \mu_3 \dots \mu_n$ is such that any three consecutive terms are related by the equation $\mu_{n+3} = 6\mu_{n+2} - 5\mu_{n+1}$. It is given that $\mu_1 = 2$ and $\mu_2 = 6$. **3**

Use mathematical induction to prove that $\mu_n = 5^{n-1} + 1$.

- (c) The roots of the equation $x^3 + 2x - 8 = 0$ are α, β and γ . Find the polynomial equation whose roots are given by
- (i) $1 - \alpha, 1 - \beta$ and $1 - \gamma$. **3**
- (ii) $\frac{\alpha + \beta}{\gamma}, \frac{\beta + \gamma}{\alpha}$ and $\frac{\alpha + \gamma}{\beta}$. **2**

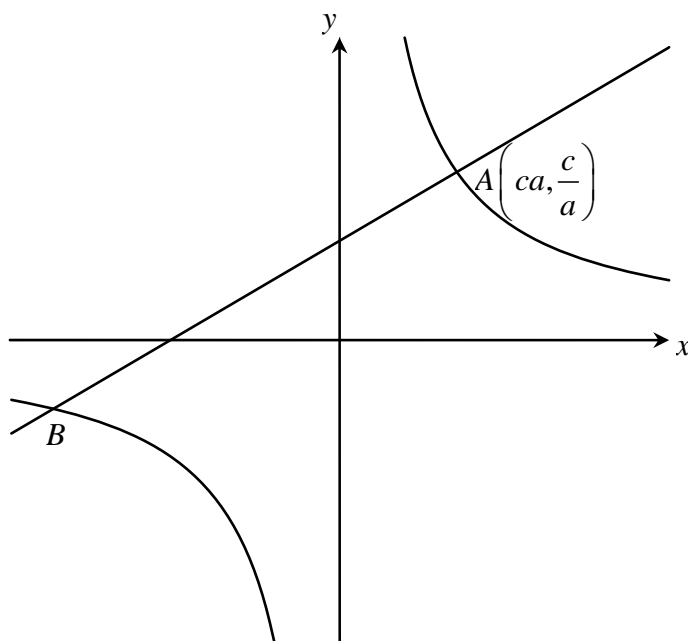
[In part (ii) consider the relationship between the coefficients and the roots]

Question 5 (15 marks) Use a SEPARATE writing booklet.

Marks

- (a) Five lines are drawn in a plane. No two lines are parallel and no three lines are concurrent.
- (i) Show that there are 10 points of intersection giving a reason for your answer. **1**
- (ii) If three of the points are chosen at random, find the probability that they all lie on one of the given lines **2**

- (b) Consider the rectangular hyperbola with equation $xy = c^2$ with points A and B which is shown below. The normal through A on the hyperbola meets the other branch at B .

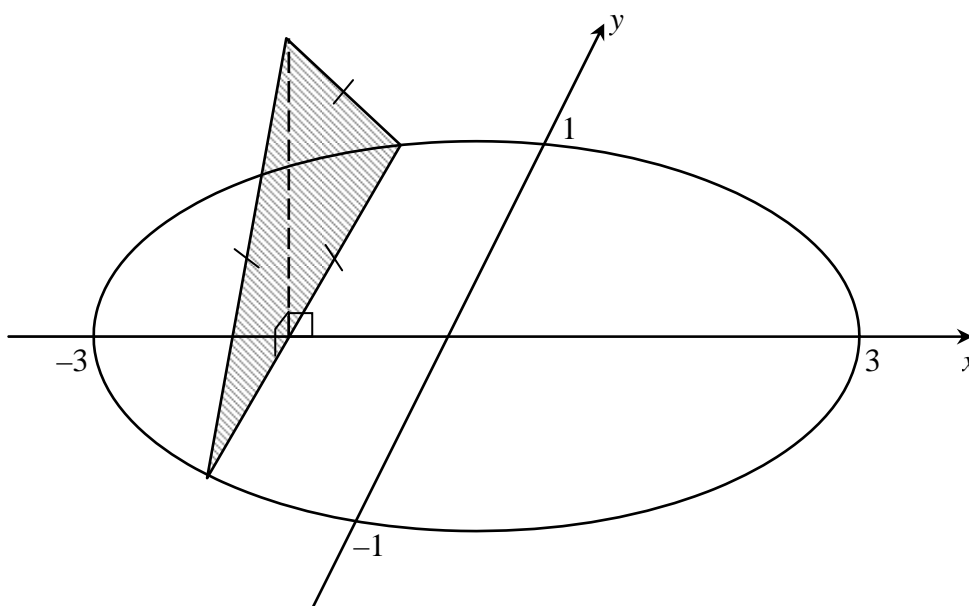


- (i) Show that the equation of the normal is given by **2**
- $$y = a^2x + \frac{c}{a}(1 - a^4)$$
- (ii) If B has co-ordinates $\left(cb, \frac{c}{b}\right)$, show that $b = -\frac{1}{a^3}$ **3**
- (iii) If this hyperbola is rotated clockwise through 45° , show that the equation becomes $x^2 - y^2 = 2c^2$. **3**
- (c) If x and y are positive numbers such that $x + y = 1$, prove that
- (i) $\frac{1}{x} + \frac{1}{y} \geq 4$ **2**
- (ii) $x^2 + y^2 \geq \frac{1}{2}$ **2**

Question 6 (15 marks) Use a SEPARATE writing booklet.

Marks

- (a) A solid shape has an elliptical base on the xy -plane as shown below. Sections of the solid taken perpendicular to the x -axis are equilateral triangles. The major and minor axes of the ellipse are of lengths 6 metres and 2 metres respectively.



- (i) Write down the equation of the ellipse. 1
- (ii) Show that the volume ΔV of a slice taken at $x = d$ is given by 2
- $$\Delta V \doteq \frac{\sqrt{3}(9-d^2)}{9} \Delta x$$
- (iii) Find the volume of this solid. 3
- (b) (i) Use graphs, or otherwise, to show that $\log_e(1+x) < x$ for $x > 0$. 2
- (ii) Sketch the graphs of $y = \frac{x}{1+x}$ and $y = \log_e(1+x)$ on the same set of axes and explain why $\frac{x}{1+x} < \log_e(1+x)$ for $x > 0$. 3
- (iii) Using the inequalities in part (i) and (ii), show that 4

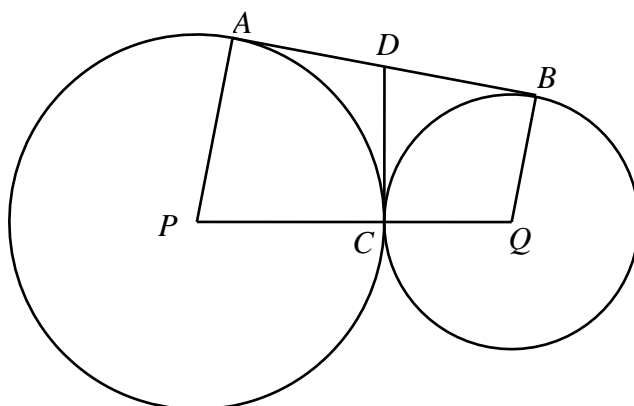
$$\frac{\pi}{8} - \frac{1}{4} \log_e 2 < \int_0^1 \frac{\log_e(1+x)}{1+x^2} dx < \log_e \sqrt{2}$$

You may assume that $\int \frac{x dx}{(1+x)(1+x^2)} = \frac{\tan^{-1} x}{2} + \frac{\ln(x^2+1)}{4} - \frac{\ln(x+1)}{2}$.

Question 7 (15 marks) Use a SEPARATE writing booklet.

Marks

- (a) In the diagram, PCQ is a straight line joining P and Q , the centres of the circles. AB and DC are common tangents.

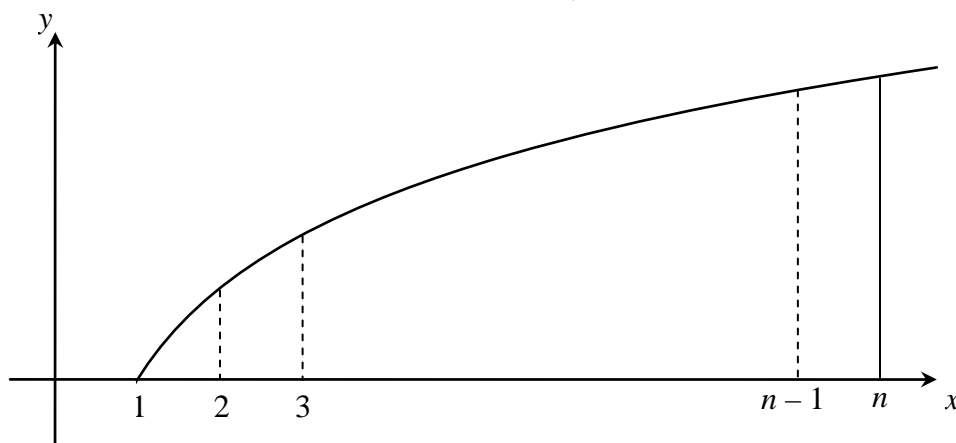


- (i) Copy the diagram into your answer booklet.
- (ii) Explain why $PADC$ and $CDBQ$ are cyclic quadrilaterals. **2**
- (iii) Show that $\triangle ADC$ is similar to $\triangle BQC$. **2**
- (iv) Show that PD is parallel to CB . **2**
- (b) (i) Sketch the region which is enclosed by the curve $y = 8x - x^2$ and the lines $x = 2$ and $x = 4$. **1**
- (ii) This region is rotated about the y -axis to generate a solid. Represent this situation on a number plane and use the method of cylindrical shells to find the volume of the solid formed. **3**
- (c) (i) Show that the function $G(x)$ where $G(x) = \frac{1}{2}[f(x) + f(-x)]$ is even **2**
and that the function $H(x)$ where $H(x) = \frac{1}{2}[f(x) - f(-x)]$ is odd.
- (ii) Deduce that the function $f(x)$ can be written as the sum of an even function and an odd function. **1**
- (iii) If $f(x) = 2^x + \tan x$, express $f(x)$ as the sum of an even function and an odd function. **2**

Question 8 (15 marks) Use a SEPARATE writing booklet.

Marks

- (a) The sketch below shows the graph of $y = \log_e x$ for $x \geq 1$.



- (i) Explain why this curve is always concave downwards. **1**
- (ii) Show that the area under $y = \log_e x$ from $x = 1$ to $x = n$ is given by $n \ln n - n + 1$ **1**
- (iii) By adding the areas of the trapezia from $x = 1$ to $x = n$, show that **3**
- $$n! < \frac{en^{\frac{1}{2}}}{e^n}$$
- (b) $P(x)$ is a polynomial of degree at least 2, such that $P'(a) = 0$. Show that when $P(x)$ is divided by $(x - a)^2$ the remainder is $P(a)$. **3**

- (c) Use mathematical induction to show that **3**

$$1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + n \times n! = (n + 1)! - 1 \text{ for } n \geq 1$$

- (d) (i) Show that $\operatorname{cosec} 2\theta + \cot 2\theta = \cot \theta$. **2**

- (ii) Hence find the value of $\cot \frac{\pi}{8}$ and $\cot \frac{\pi}{12}$ and show that **2**

$$\operatorname{cosec} \frac{2\pi}{15} + \operatorname{cosec} \frac{4\pi}{15} + \operatorname{cosec} \frac{8\pi}{15} + \operatorname{cosec} \frac{16\pi}{15} = 0$$

End of paper

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

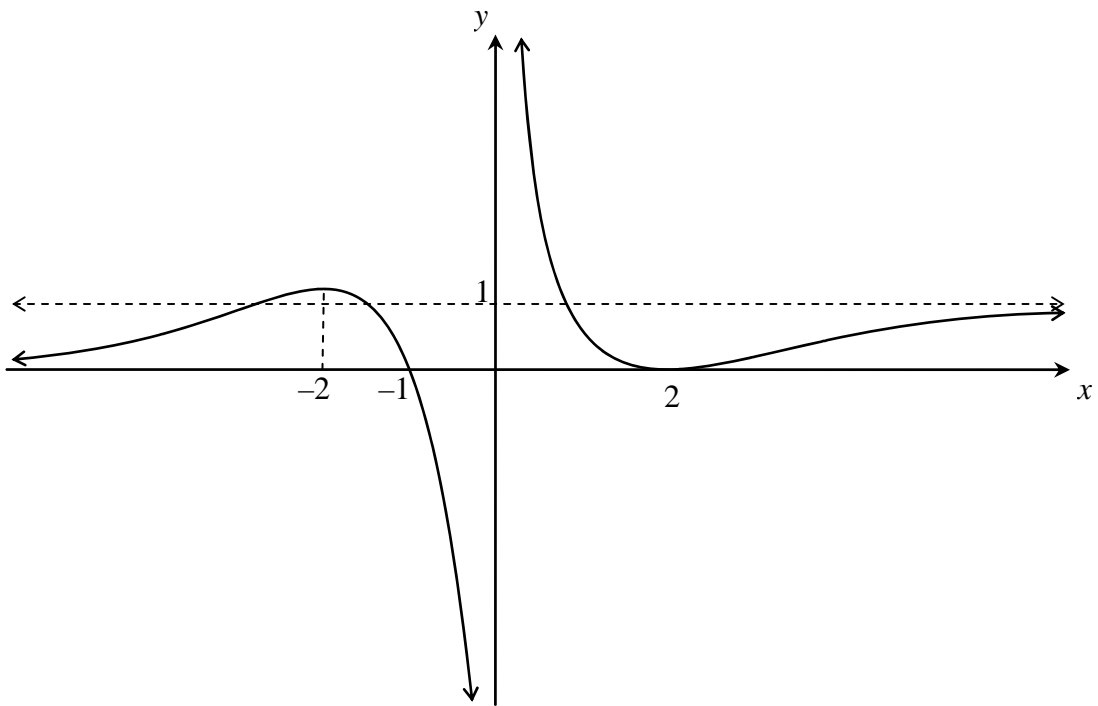
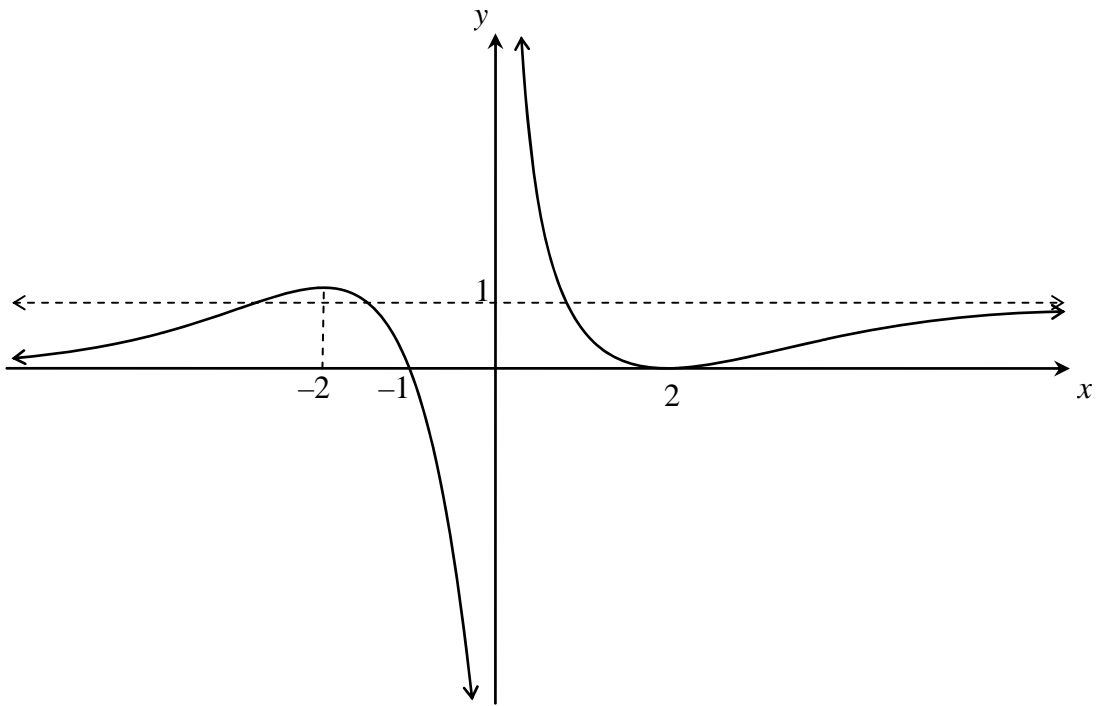
$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

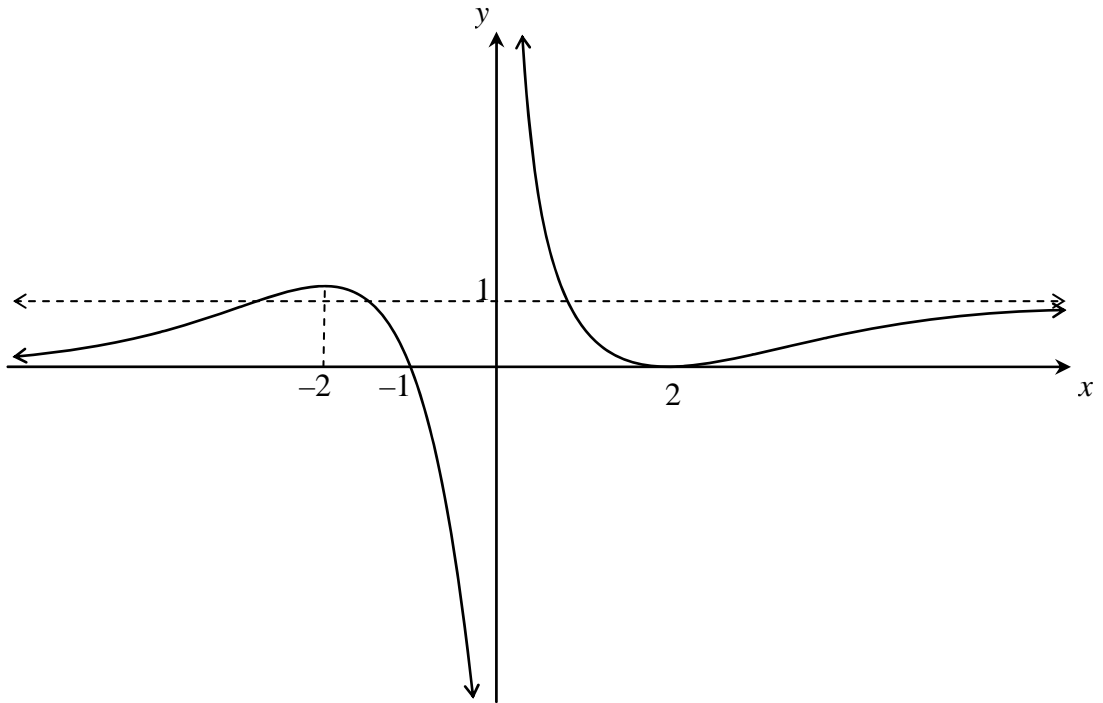
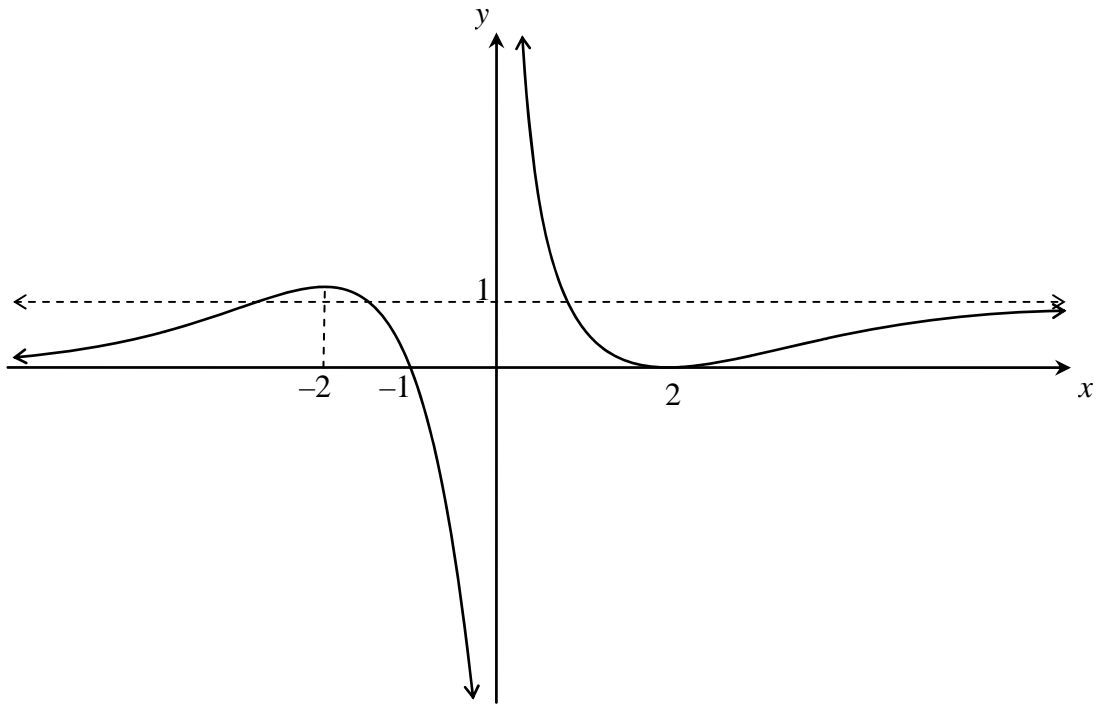
NOTE : $\ln x = \log_e x, \quad x > 0$

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Use this page for your answers to Question 3a)



Insert this page in your booklet for Question 3



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