NORTH SYDNEY GIRLS HIGH SCHOOL

2009

TRIAL HIGHER SCHOOL CERTIFICATE

Mathematics Extension 2

Student Number:________________________________**Teacher:** _______________

Student Name:__________________________________

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- All *necessary* working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for untidy or badly arranged work.
- Start each **NEW** question in a separate answer booklet.

Total Marks - 120 Marks

- Attempt Questions 1-8
- All questions are of equal value.

At the end of the examination, place your solution booklets in order and put this question paper on top.
Submit one bundle.
The bundle will be separated before marking commences so that anonymity will be maintained.

Question	1	2	3	4	5	6	7	8	Total
Mark									/120

Total marks – 120
Attempt Questions 1 - 8
All questions are of equal value

Answer each section in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find $\int \cos x \sin^6 x \, dx$ **1**

(b) Evaluate $\int_0^1 \frac{2+6x}{\sqrt{4-x^2}} \, dx$, leaving your answer in exact form **3**

(c) Use integration by parts to evaluate $\int_0^{\sqrt{3}} x \tan^{-1} x \, dx$ **4**

(d) (i) Find real constants A and B such that $\frac{3}{(x-2)(2x-1)} = \frac{A}{x-2} + \frac{B}{2x-1}$ **2**

(ii) Hence find $\int \frac{3\,dx}{(x-2)(2x-1)}$ **2**

(e) Using the substitution $t = \tan\frac{\theta}{2}$ and the results of (d), find **3**

$$\int \frac{5}{4-3\sin\theta} \, d\theta$$

Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $z=\dfrac{3-6i}{2+i}$, find

(i) $|z|$ **2**

(ii) $\arg z$ **2**

(b) Find real values p and q where $\dfrac{p-5qi}{1+i}=\overline{1-4i}$ **3**

(c) Let $u=\dfrac{7\sqrt{2}}{2}(1+i)$, $v=r\cos\theta+ir\sin\theta$ and $uv=42\left(\cos\dfrac{\pi}{20}+i\sin\dfrac{\pi}{20}\right)$

(i) Write u in modulus-argument form. **2**

(ii) Find r and θ. **2**

(d) z lies on the locus defined by $|z+2|=2$ and let $\arg z=\theta$

(i) By use of an appropriate diagram, show that $\arg(z+2)=2\theta-\pi$ **2**

(ii) Hence, or otherwise, find $\arg(z^2+6z+8)$ **2**

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Consider the rectangular hyperbola $x^2 - y^2 = 4$.

(i) Sketch the curve, showing the coordinates of the foci S and S' and the equations of the directrices and asymptotes. **3**

(ii) The point $P(2\sec\theta, 2\tan\theta)$ lies on the curve. **2**

Show that the tangent at P has equation $x\sec\theta - y\tan\theta = 2$.

(iii) The tangent meets the x-axis at Q. **3**

Show that the locus of the midpoint M of PQ is given by $x^2 - y^2 - 3 = \dfrac{1}{y^2+1}$

(b) The polynomial $u(x) = mx^7 + nx^6 + 1$ is divisible by $(x+1)^2$.

(i) Show that $7m = 6n$. **1**

(ii) Find the values of m and n, where m and n are real numbers. **2**

(c) Given that α, β and γ are the roots of the equation $x^3 + 3x + 1 = 0$

(i) Find a polynomial equation of smallest degree that has α^2, β^2 and γ^2 as roots. **2**

(ii) Hence find $\alpha^2 + \beta^2 + \gamma^2$ **1**

(d) Which one of the following diagrams below could represent the location of the roots of $z^5 + z^2 - z + c = 0$ in the complex plane, where c is a real number. Without any calculations, justify your answer. **1**

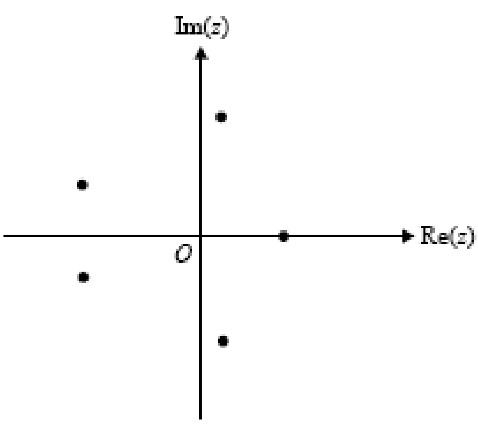


Diagram A

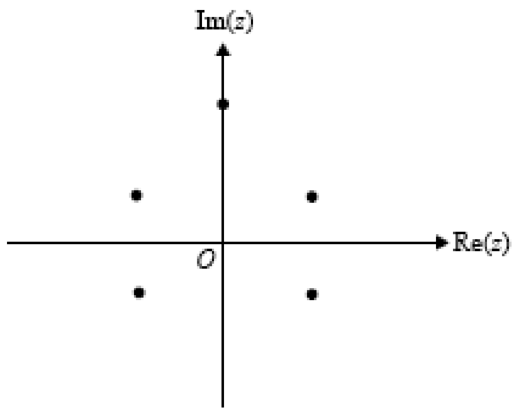
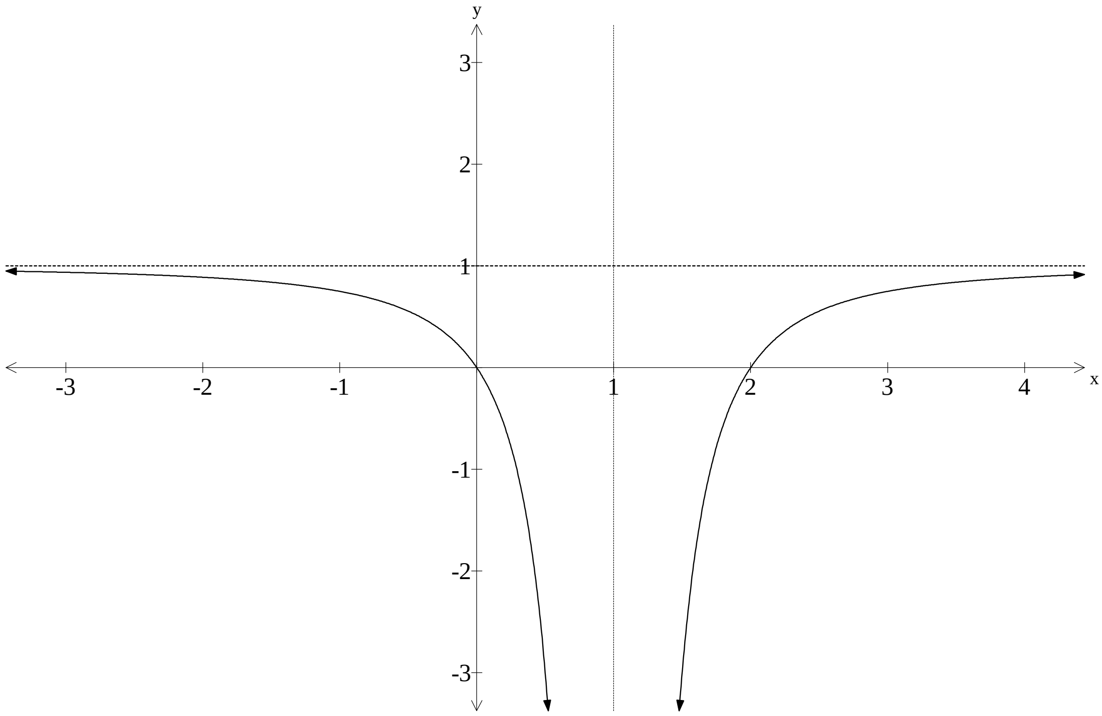


Diagram B

Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The graph below shows a function that has x-intercepts at $x = 0$ and $x = 2$. There is a vertical asymptote at $x = 1$ and a horizontal asymptote of $y = 1$. The graph is symmetrical about the line $x = 1$.



Without using calculus, sketch the following graphs on the ANSWER sheet provided on page 15, clearly showing any asymptotes and intercepts.

(i) $y = f(x-1)$ **1**

(ii) $y = \left[f(x)\right]^2$ **2**

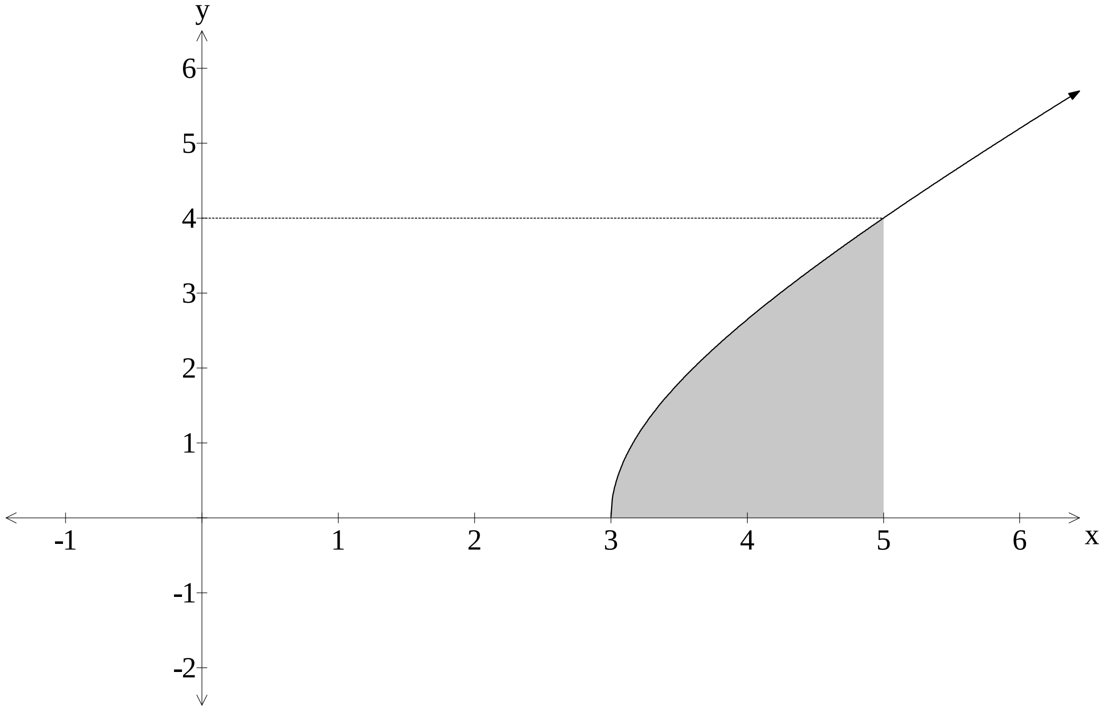
(iii) $y^2 = f(x)$ **2**

(iv) $y = \tan^{-1} f(x)$ **2**

Question 4 continues on page 6

(b) The graph of $f(x)=\sqrt{x^2-9}$ is shown below.

The area between $f(x)$ and the x-axis for $3 \le x \le 5$ is shaded.



(i) Using the method of shells, show that the volume, V, of the solid formed when the shaded area is rotated about the y-axis is given by **2**

$$V=\int_3^5 2\pi x\left(x^2-9\right)^{\frac{1}{2}}\,dx$$

(ii) Hence calculate the volume. **1**

(c) (i) Given $f(x)=f(a-x)$ and using the substitution $u=a-x$, prove that **3**

$$\int_0^a xf(x)\,dx=\frac{a}{2}\int_0^a f(x)\,dx$$

(ii) Hence, or otherwise, prove that $\displaystyle\int_0^\pi F(x)\,dx=\frac{\pi^2}{4}$, if $F(x)=\dfrac{x\sin x}{1+\cos^2 x}$ **2**

Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Given $I_n = \int_0^1 x^n e^{2x}\, dx$, where n is a positive integer, show that **2**

$$I_n = \frac{1}{2}\left(e^2 - nI_{n-1}\right)$$

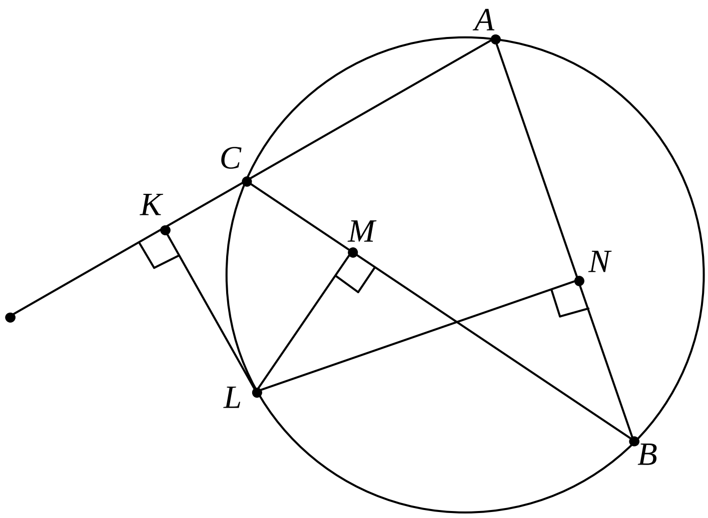
(ii) Hence evaluate $\int_0^1 x^3 e^{2x}\, dx$ **3**

(b) Fifteen new students at NSGHS are distributed evenly among the classes of Miss V, Mr. S and Ms L.
Given that there are three children with red hair among the fifteen and that the students are distributed randomly, find:

(i) the number of ways that all the children with red hair end up in the same class. **2**

(ii) the probability that each class gets one child with red hair. **2**

(c) The diagram below shows triangle ABC inscribed in a circle with L a point on the arc BC.
LK is perpendicular to AC produced and LN is perpendicular to AB.



(i) Copy the diagram into your Answer book

(ii) Explain why $CKLM$ and $MNBL$ are cyclic quadrilaterals. **2**

(iii) Explain why $\angle KCL = \angle ABL$. **1**

(iv) Hence, or otherwise, prove that K, M and N are collinear. **3**

Question 6 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Solve $\sin^{-1} x - \cos^{-1} x = \sin^{-1}(3x-2)$ **4**

(b) Given that $\sin\left(\frac{1}{2}y\right) = \frac{1}{2}\left(x^2 - 2\right)$ and that $x > 0$ and $y > 0$. **3**

Show by differentiating implicitly that $\dfrac{dy}{dx} = \dfrac{4}{\sqrt{4-x^2}}$

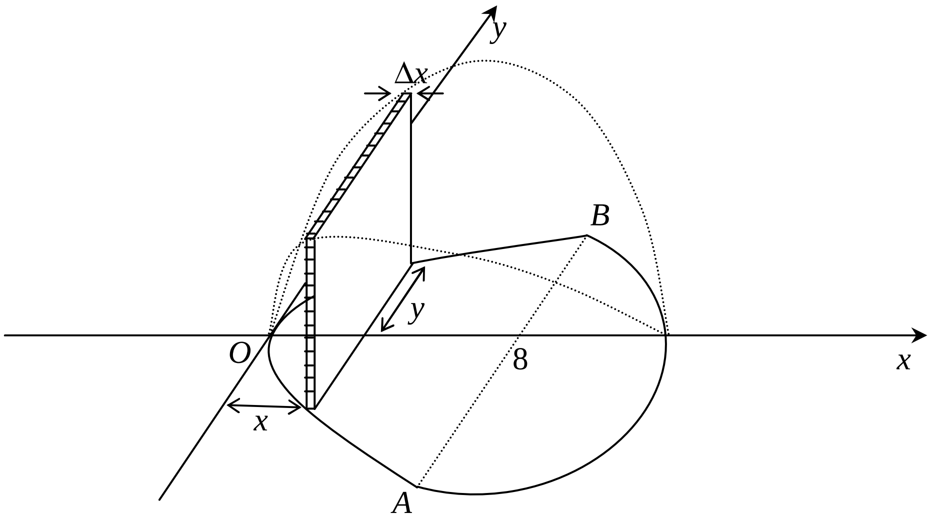
(c) The diagram below shows a solid with its base in the *x*-*y* plane.

Every cross-section perpendicular to the *x*-axis is a square.

One part of the base is the segment *OAB* of the parabola $y^2 = 2x$ cut off by the line $x = 8$.

The other part of the base is a semi-circle with diameter *AB*.

Consider a slice *S*, perpendicular to the *x*-axis, of width Δx.



(i) Find the coordinates of *B* and hence find the distance *AB*. **2**

(ii) Show that the volume ΔV of *S* is given by $\Delta V \approx 8x\Delta x$ for $0 \le x \le 8$. **2**

(iii) By first finding an expression for ΔV of *S* when $x > 8$, calculate the volume of the solid. **4**

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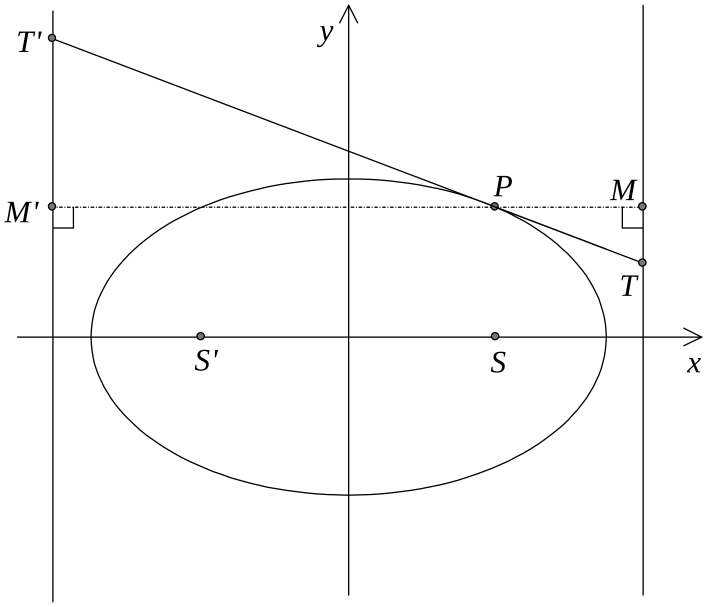
Question 7 (15 marks) Use a SEPARATE writing booklet. **Marks**

The diagram below shows an ellipse $b^2x^2 + a^2y^2 = a^2b^2$, where S and S' are the foci.

The diagram shows a tangent at $P\left(a\cos\theta, b\sin\theta\right)$, intersecting the two directrices at T and T'.

M and M' are the foot of the perpendiculars drawn from P to their respective directrices.



(a) Show that $SP + S'P = 2a$. **2**

(b) You may assume that the tangent at P is $xb\cos\theta + ya\sin\theta = ab$. (Do **NOT** prove this)

Let $\alpha = \angle SPT$ and $\beta = \angle S'PT'$

(i) Show that T has coordinates $\left(\dfrac{a}{e}, \dfrac{b\left(e - \cos\theta\right)}{ae\sin\theta}\right)$ **1**

(ii) Show that $\angle PST = 90°$ **2**

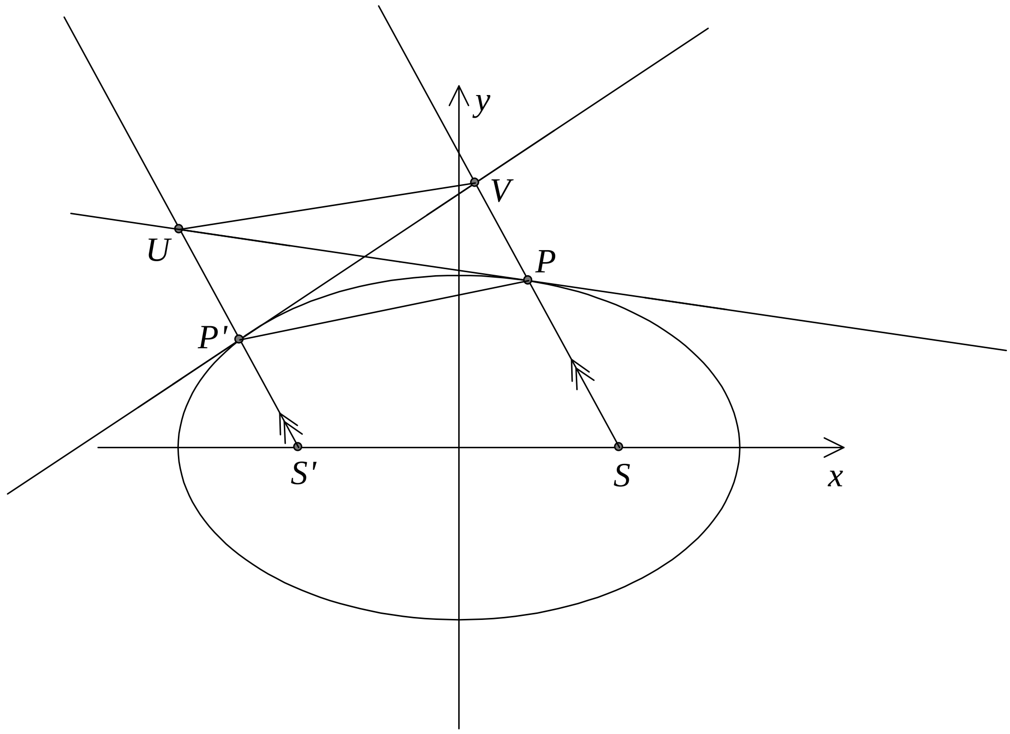
(iii) Show that $\dfrac{PM}{PT} = \dfrac{PM'}{PT'}$ **1**

(iv) Deduce that $\alpha = \beta$. **2**

Question 7 continues on page 11

(c) Consider the diagram below, where $SV \parallel S'U$.

The tangent at P intersects the ray $S'U$ at U and the tangent at P' intersects the ray SV at V.



(i) Copy the diagram into your Answer booklet.

(ii) Using (b) show that $\Delta UPS'$ is isosceles. **3**

(iii) Using (ii) above and also (a), show that $VP = UP'$. **3**

(iv) Deduce that $UV \parallel PP'$ **1**

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Show that $\tan^{-1}(n+1)-\tan^{-1}(n-1)=\tan^{-1}\left(\frac{2}{n^2}\right)$, where n is a positive integer. **2**

(ii) Given that $\tan\left(\tan^{-1}x+\tan^{-1}y\right)=\frac{x+y}{1-xy}$, where x and y are real numbers, explain why when $x>1,\ y>1$ that $\tan^{-1}x+\tan^{-1}y=\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right)$. **1**

(iii) Hence, or otherwise, show that for $n\geq 1$ **3**

$$\sum_{r=1}^{n}\tan^{-1}\left(\frac{2}{r^2}\right)=\frac{3\pi}{4}+\tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right)$$

(iv) Hence write down $\sum_{r=1}^{\infty}\tan^{-1}\left(\frac{2}{r^2}\right)$ **1**

(b) Let $T_n(x)=\frac{{}^nC_0}{x}-\frac{{}^nC_1}{x+1}+\frac{{}^nC_2}{x+2}-\quad.......\quad+(-1)^n\frac{{}^nC_n}{x+n}$ for a given integer n and all real x

(i) If $S_k(x)=\frac{k!}{x(x+1)(x+2)....(x+k)}$ where k is an integer, show that **2**

$$S_k(x)-S_k(x+1)=S_{k+1}(x)$$

(ii) Hence prove using mathematical induction that for $n\geq 1$ **4**

$$T_n(x)=\frac{n!}{x(x+1)(x+2)....(x+n)}$$

NOTE: you may use without proof the result ${}^{m+1}C_r={}^mC_r+{}^mC_{r-1}$

(iii) Hence by a suitable substitution prove that **2**

$$\frac{{}^nC_0}{1}-\frac{{}^nC_1}{3}+\frac{{}^nC_2}{5}-\quad...\quad+(-1)^n\frac{{}^nC_n}{2n+1}=\frac{2^n n!}{1\times 3\times 5\times...\times(2n+1)}$$

End of paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

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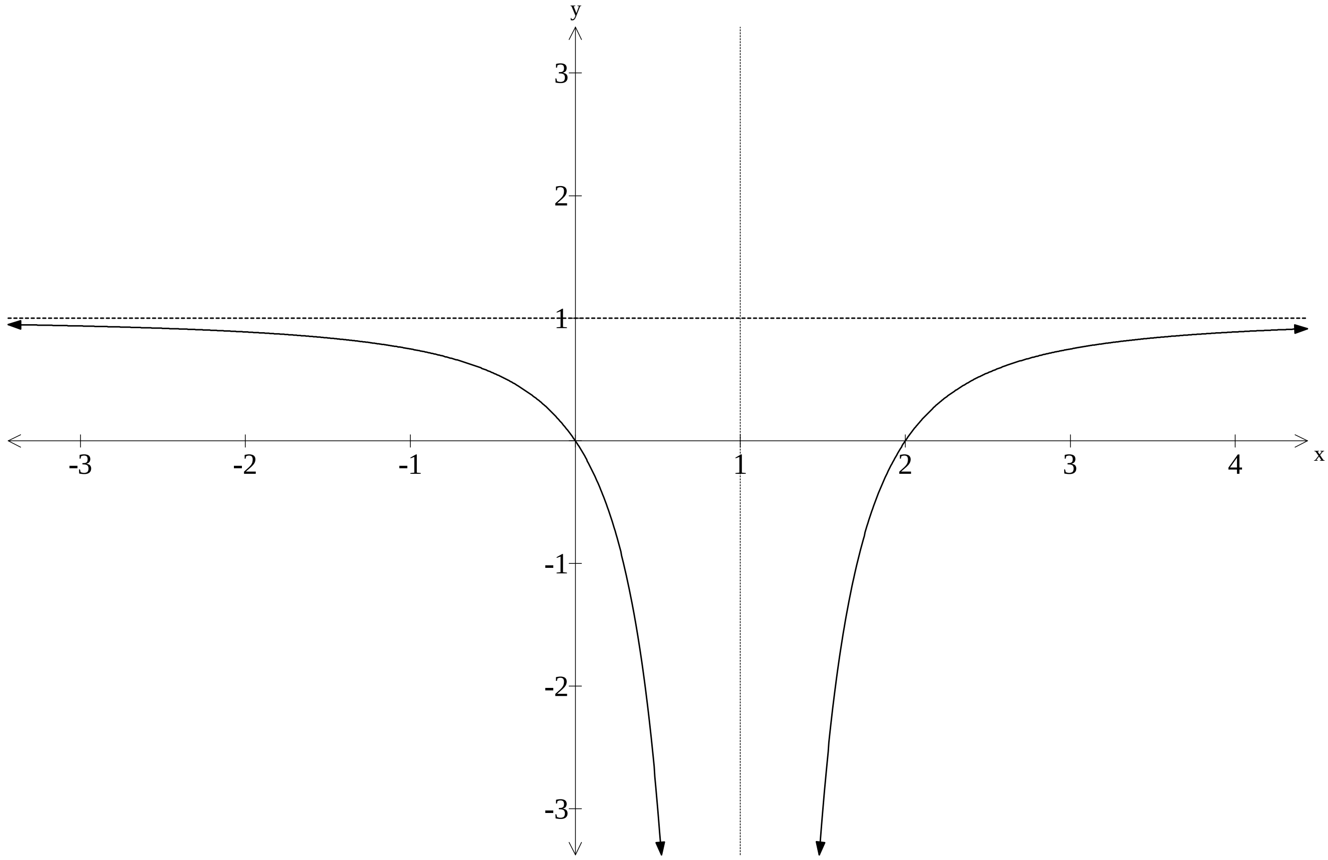
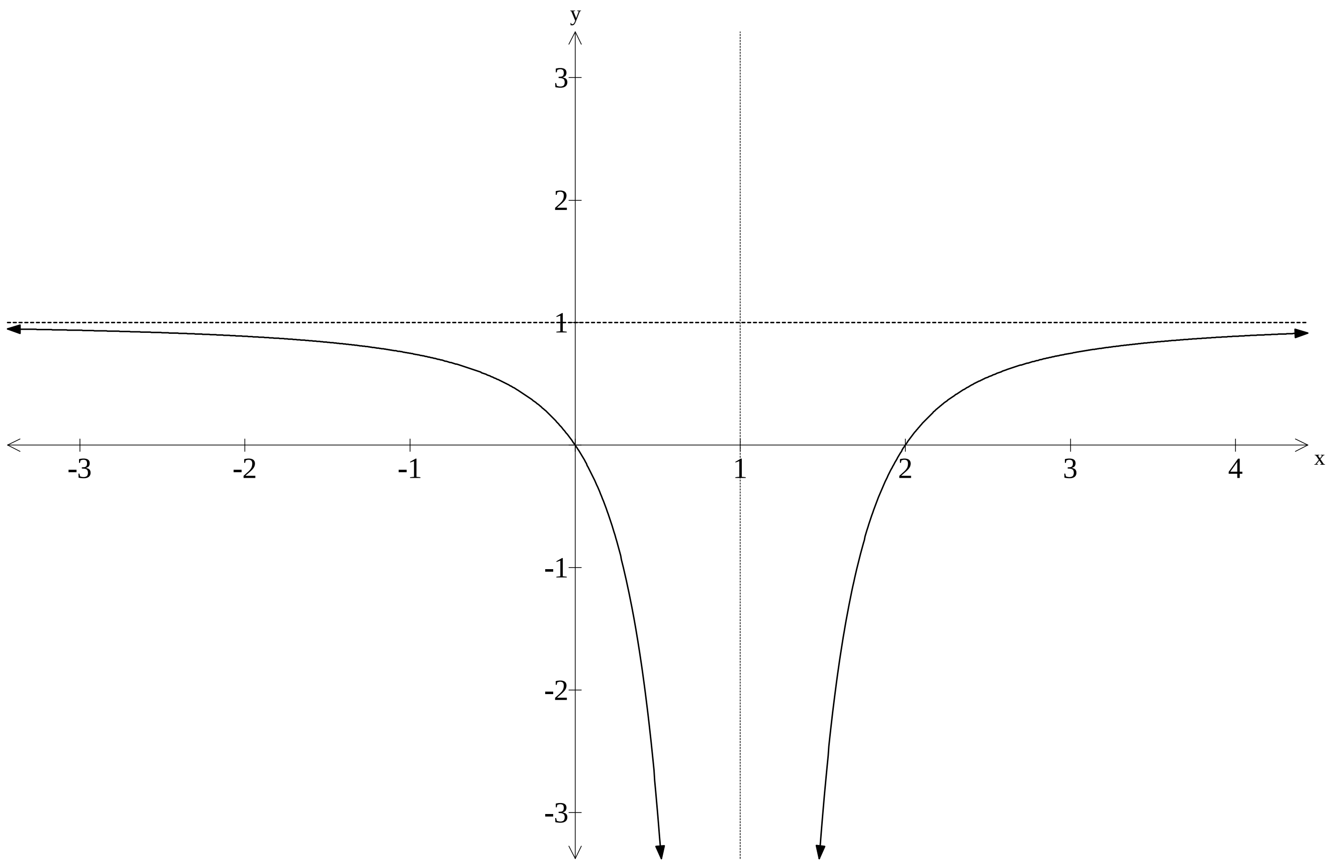
NSGHS 2009 Trial HSC Extension 2 Mathematics Exam Answer Sheet for Question 4 (a)

Number______________________________ **Teacher:**______________________

(i)



(ii)



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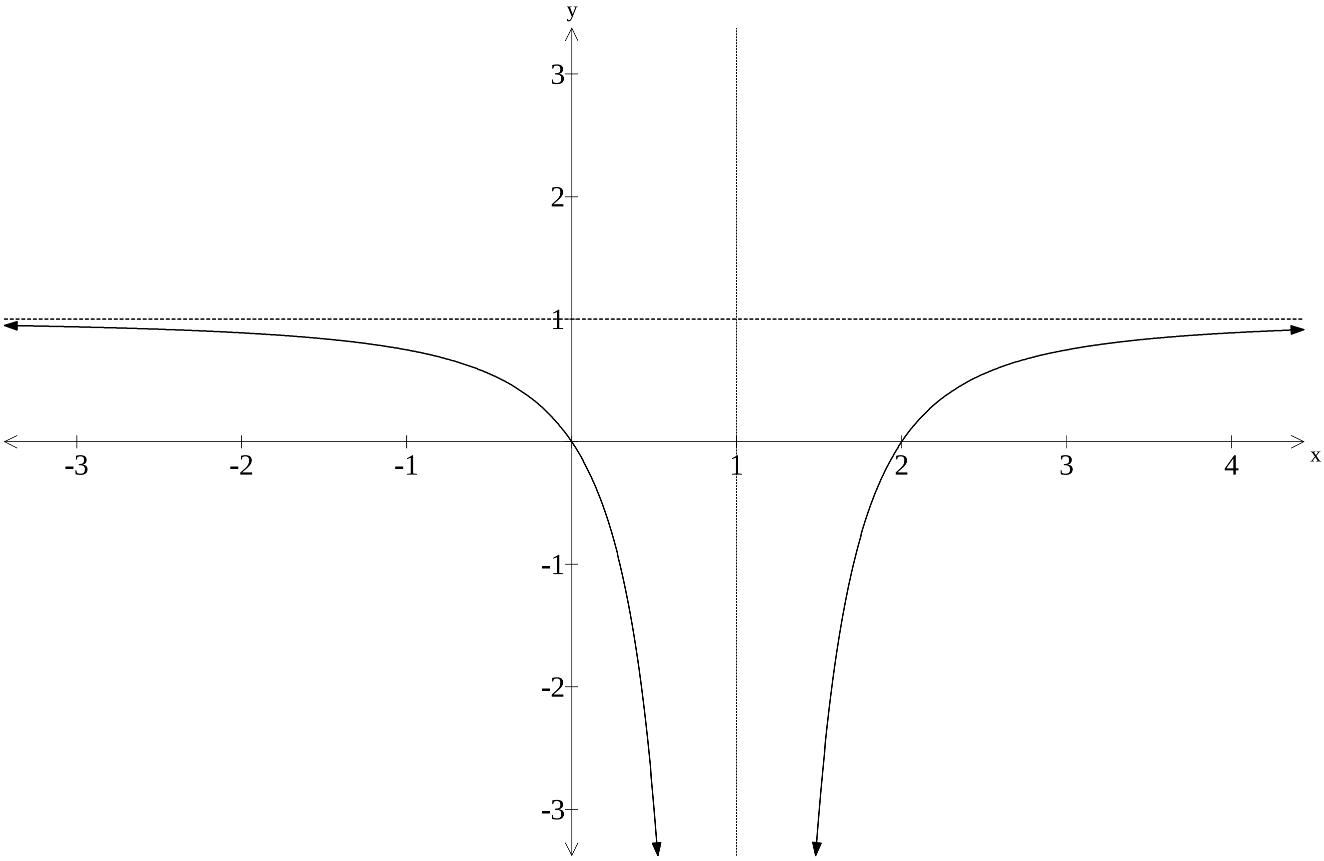
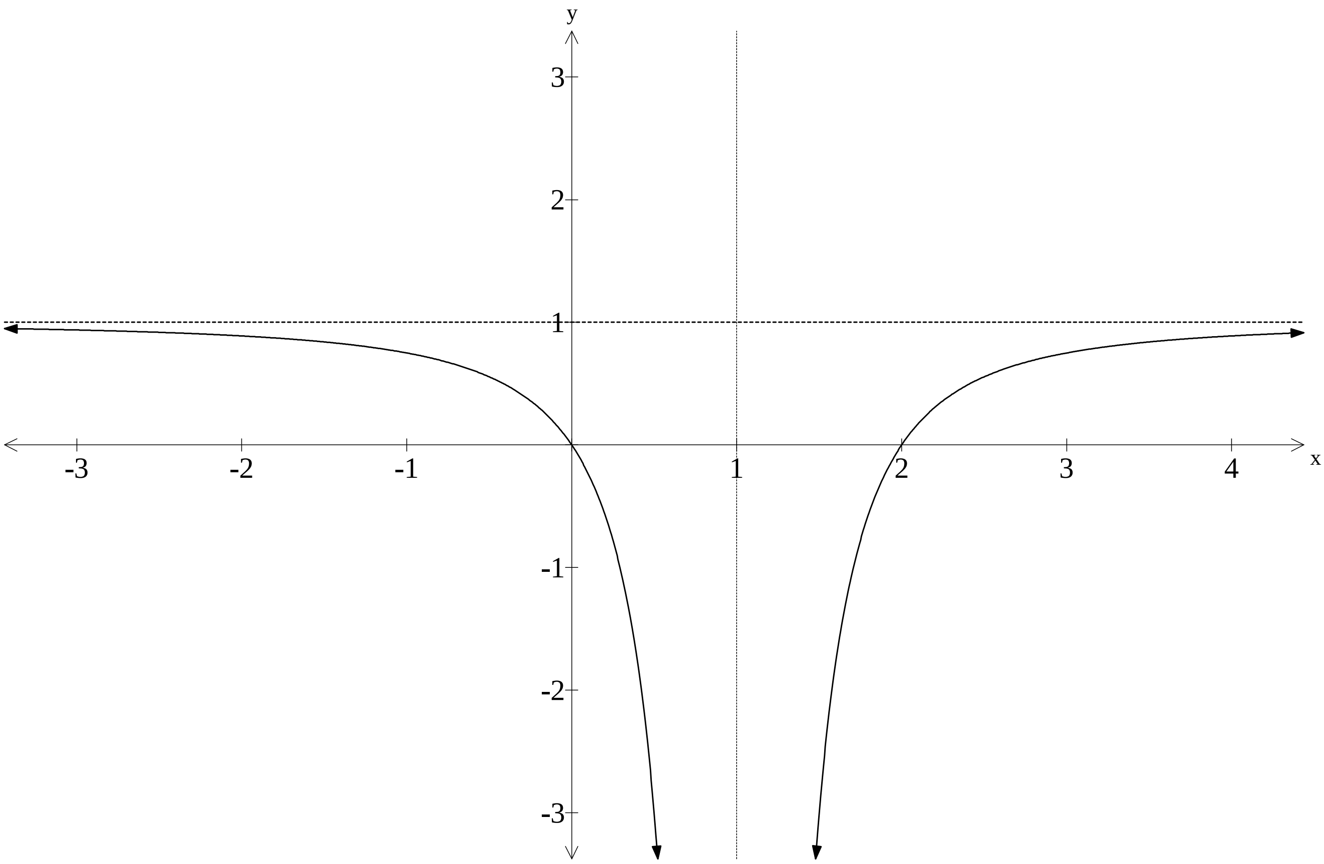
Answer Sheet for Question 4 (a) continued

(iii)



(iv)



NOW PLACE THIS SHEET INSIDE YOUR BOOKLET FOR QUESTION 4