NORTH SYDNEY GIRLS HIGH SCHOOL

2006
TRIAL HSC EXAMINATION

Mathematics Extension 2

General Instructions
- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black or blue pen
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this page

- All necessary working should be shown in every question

Total Marks – 120
Attempt Questions 1–8
All questions are of equal value

NAME:_____   **TEACHER:** _____

NUMBER:_____

QUESTION	MARK
1	/15
2	/15
3	/15
4	/15
5	/15
6	/15
7	/15
8	/15
TOTAL	/120

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Attempt Questions 1–8
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Begin each question in a NEW BOOKLET.

Question 1 (15 marks)   **Marks**

a) Evaluate

    i) $\displaystyle \int_0^{\frac{\pi}{4}} \frac{\cos x}{1+\sin^2 x}\,dx$.   **2**

    ii) $\displaystyle \int_0^{\sqrt{2}} \sqrt{4-x^2}\,dx$ using the substitution $x = 2\sin\theta$.   **3**

b) Find $\displaystyle \int x^2 e^x\,dx$.   **2**

c) i)   Write $\displaystyle \frac{4x^2+11x-8}{(x+2)(x^2-x+1)}$ in the form $\displaystyle \frac{A}{x+2}+\frac{Bx+C}{x^2-x+1}$ **2**

    ii)   Hence evaluate $\displaystyle \int_{-1}^{0} \frac{4x^2+11x-8}{(x+2)(x^2-x+1)}\,dx$.   **2**

d) i)   Prove that $\displaystyle \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.   **2**

    ii)   Hence find $\displaystyle \int_0^1 x(1-x)^{99}\,dx$.   **2**

Question 2: Begin a new booklet (15 marks) **Marks**

a) If $z = 2 + 3i$ and $w = 1 - i$, find each of the following, expressing your answer in the form $x + iy$:

   i) $z + \overline{w}$ **1**

   ii) $\dfrac{z - 2}{w}$ **1**

b) Write $-\sqrt{3} + i$ in modulus-argument form and hence evaluate $(-\sqrt{3} + i)^{12}$. **2**

c) i) On the Argand diagram, clearly indicate the region containing all points representing the complex number z which satisfies the following conditions:

   $$0 \le \arg\,[z - (1 + i)] \le \frac{3\pi}{4}\ ,\ |z - 1| \le |z - 3|\ \text{ and }\ \operatorname{Re} z \ge 0$$ **4**

   ii) Hence find the range of values for $\arg z$ when z lies in the region shaded in i) above. **1**

d) Let z be a complex number such that $|z| = 1$ and $\arg z = \theta$, where $0 \le \theta \le \dfrac{\pi}{2}$.

   i) On an Argand diagram, illustrate the points P and Q which represent z and z^2 respectively, clearly indicating their relationship. **2**

   ii) Evaluate, in terms of θ,

   $\alpha)\quad \left|\dfrac{2}{1 - z^2}\right|$ **1**

   $\beta)\quad \arg\left(\dfrac{2}{1 - z}\right)$ **1**

e) Suppose that w is the complex number $a + ib$. For what conditions on a and b is $w + \dfrac{1}{w}$ purely real? **2**

Question 3: Begin a new booklet (15 marks)　　　　　　　　　　　**Marks**

a)　　The graph of $y = f(x)$ is illustrated. The line $y = 0$ is a horizontal asymptote
　　　　and $x = 5$ is a vertical asymptote.
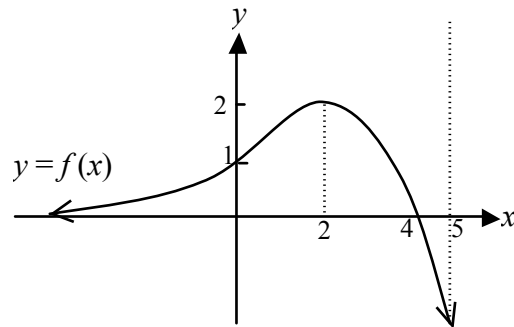
　　　　Using the separate page of graphs provided, sketch each of the graphs below.
　　　　In each case, clearly label any maxima or minima, intercepts and the equations
　　　　of any asymptotes.

　　　　i)　　　$y = \dfrac{1}{f(x)}$　　　　　　　　　　　　　　　　　　　　　　　　2

　　　　ii)　　$y^2 = f(x)$　　　　　　　　　　　　　　　　　　　　　　　　　　2

　　　　iii)　　$y = f(|x|)$　　　　　　　　　　　　　　　　　　　　　　　　　　2

　　　　iv)　　$y = [f(x)]^3$　　　　　　　　　　　　　　　　　　　　　　　　　2

　　　　v)　　　$y = e^{f(x)}$　　　　　　　　　　　　　　　　　　　　　　　　　2

b)　　Consider the function $f(x) = x - \ln(x^2 + 1)$ for $x \geq 0$.
　　　　i)　　　Show that $f'(x) \geq 0$ for $x \geq 0$　　　　　　　　　　　　　　1
　　　　ii)　　Hence deduce that $x > \ln(x^2 + 1)$ for $x > 0$.　　　　　　　　　1

c)　　Two sides of a triangle are in the ratio 3:1 and the angles opposite these
　　　　sides differ by $\dfrac{\pi}{6}$. Show that the smaller of the two angles is $\tan^{-1}\left(\dfrac{1}{6 - \sqrt{3}}\right)$.　　3

Question 4: Begin a new booklet (15 marks)   **Marks**

a)   The area enclosed by the curve $xy = 4$ and the line $y = 5 - x$ is rotated about
        the line $y = 5$ to form a solid.

        i)   Draw a diagram to illustrate the region.   **1**

        ii)   By taking slices of the solid perpendicular to the axis of rotation,
                show that the volume of the solid is given by

                $$V = \pi \int_1^4 \left\{ \left(5 - \frac{4}{x} \right)^2 - x^2 \right\} dx$$ **3**

        iii)   Hence, find the volume of the solid correct to 2 significant figures.   **2**

b)   i)   Prove by Mathematical Induction that if n is a positive integer, then
                $2^{(n+4)} > (n+4)^2$.   **3**

        ii)   By choosing a suitable substitution, or otherwise, show that if a is a
                positive integer, then $2^{3(a+4)} > 9(a+4)^2$.   **1**

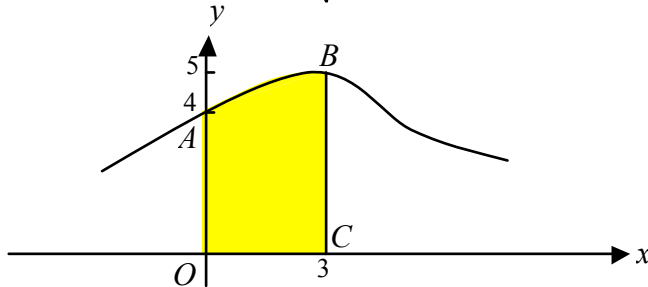c)   i)   If $I_n = \int \tan^n x \sec x \, dx$ for integral values of $n \geq 0$, show that
                $nI_n = \tan^{n-1}x \sec x - (n-1)I_{n-2}$ for $n \geq 2$.   **3**

        ii)   Hence evaluate $\int_0^{\frac{\pi}{4}} \frac{\sin^3 x}{\cos^4 x} dx$.   **2**

Question 5: Begin a new booklet (15 marks)   **Marks**

a)   The diagram shows part of the curve whose parametric equations are given

by $x = t + 3$ and $y = \dfrac{20}{\sqrt{t^2 + 16}}$.

i)   Find the values of t that correspond to each of the points A and B
        on the curve.   1

ii)   A solid is formed by rotating the region $OABC$ about the y-axis.
        Use the method of cylindrical shells to express the volume in the form

        $V = 40\pi \displaystyle\int_{-3}^{0} \dfrac{t+3}{\sqrt{t^2+16}}\, dt$.   3

iii)   Hence show that the volume of the solid is $40\pi(3\ln 2 - 1)$ unit^3.   3

b)   The base of a solid is the semi-circular region of radius 1 unit in the x-y plane
        as illustrated in the diagram below.
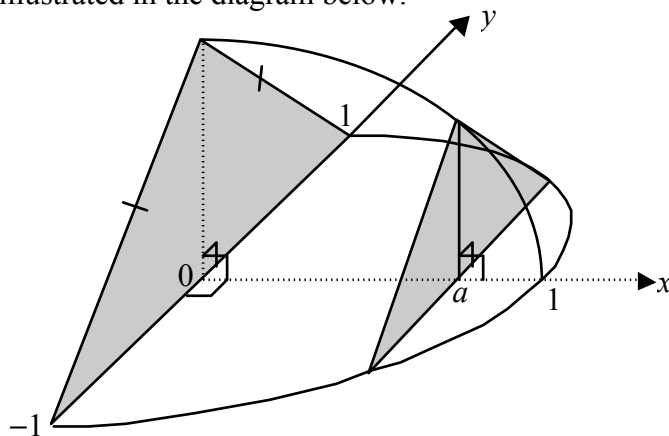
        Each cross-section perpendicular to the x-axis is an isosceles triangle with
        each of the two equal sides three quarters the length of the third side.

i)   Show that the area of the triangular cross-section at $x = a$ is $\dfrac{\sqrt{5}}{2}\left(1 - a^2\right)$.   2

ii)   Hence find the volume of the solid.   3

c)   i)   Find the sum of the series $1 + x + x^2 + x^3 + \ldots + x^n$ 1

        ii)   Hence find the sum of the series $x + 2x^2 + 3x^3 + \ldots + nx^n$ 2

Question 6: Begin a new booklet (15 marks)   **Marks**

a)   The velocity, v ms^{-1}, of a particle moving on the x-axis is given by
$$v^2 = 7 + 20x - 3x^2$$

      i)   Show that the particle is moving in simple harmonic motion and find the centre of the motion.   **2**

      ii)   Find also the amplitude and period of the motion.   **2**

b)   A particle P is thrown downwards in a medium where the resistive force is proportional to the speed.

      i)   Taking the downward direction as positive, explain why $\ddot{x} = g - kv$, where g is the acceleration due to gravity and $k > 0$.   **1**

The initial speed is U ms^{-1} and the particle is thrown from a point T which is d metres above a fixed point O, which is taken as the origin.

      ii)   Show that the velocity, v ms^{-1}, at any time, t seconds, is given by
$$v = \frac{g}{k} - \left(\frac{g - kU}{k} \right) e^{-kt}.$$ **2**

      iii)   Show that the displacement, x metres, at any time, t seconds, is given

by $$x = \frac{gt - kd}{k} + \left(\frac{g - kU}{k^2} \right)\left(e^{-kt} - 1 \right)$$ **3**

      iv)   An identical particle Q is dropped from O at the same instant that P is thrown down from T. Use the above results to write down expressions for v and x as functions of t for the particle Q.   **2**

      v)   The particles P and Q collide. Show that the speed at which the particles collide is $\left| U - kd \right|$ ms^{-1}.

(Note: the speed of collision is the difference between the two speeds.)   **3**

Question 7: Begin a new booklet (15 marks)   **Marks**

a)   The points A, B, C and D lie on the circle C_1. From the exterior point T, a tangent is
        drawn to point A on C_1. The line CT passes through D and TC is parallel to AB.
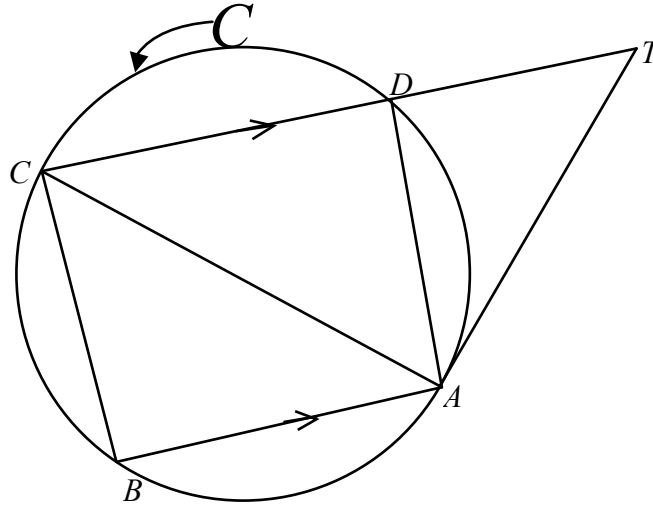
       i)   Copy or trace the diagram onto your page.

       ii)   Prove that $\triangle ADT$ is similar to $\triangle ABC$.   **3**

       The line BA is produced through A to point M, which lies on a second circle C_2.
       The points A, D, T also lie on C_2 and the line DM crosses AT at O.

       iii)   Show that $\triangle OMA$ is isosceles.   **2**

       iv)   Show that $TM = BC$.   **2**

b)   i)   Prove that the normal to the hyperbola $xy = 4$ at the point $P(2p, \dfrac{2}{p})$
               is given by $p^3x - py = 2(p^4 - 1)$.   **2**

       ii)   If the normal meets the hyperbola again at $Q(2q, \dfrac{2}{q})$ prove that
               $p^3q = -1$.   **2**

       iii)   Hence prove that there exists only one chord which is normal to the
               hyperbola at both ends and find its equation.   **4**

Question 8: Begin a new booklet (15 marks)   **Marks**

a)   i)   Find in modulus-argument form, the four roots of -16 and
           illustrate these on the Argand diagram.   **3**

     ii)   Hence or otherwise, write $z^4 + 16$ as a product of two quadratic
           factors with real coefficients.   **2**

     iii) Let α be the root of $z^4 = -16$ which has a principal argument
           between 0 and $\dfrac{\pi}{2}$. Show that $\alpha + \dfrac{\alpha^3}{4} + \dfrac{\alpha^5}{16} + \dfrac{\alpha^7}{64} = 0$.   **3**

b)   If one root of the equation $x^3 - px^2 + qx - r = 0$ is equal to the product
     of the other two roots, show that $r(p + 1)^2 = (q + r)^2$.   **3**

c)   If $f(xy) = f(x) + f(y)$ for all $x, y \neq 0$ prove that

     i) $f(x^3) = 3f(x)$ **1**

     ii) $f(1) = f(-1) = 0$ **2**

     iii) $f(x)$ is an even function   **1**

End of paper