100 Years
NORTH SYDNEY BOYS HIGH SCHOOL
1912 - 2012

MATHEMATICS (EXTENSION 2)

2012 HSC Course Assessment Task 3 (Trial Examination)
June 21, 2012

General instructions

- Working time – 3 hours.
  (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets + answer sheet used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer sheet provided (numbered as page 10)

SECTION II

- Commence each new question on a new page. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

STUDENT NUMBER: ..... **# BOOKLETS USED**: .....

Class (please ✔)

◯ 12M4A – Mr Weiss ◯ 12M4B – Mr Ireland ◯ 12M4C – Mr Fletcher

Marker's use only.

QUESTION	1-10	11	12	13	14	15	16	**Total**
MARKS	/10	/15	/15	/15	/15	/15	/15	/100

Section I: Objective response

Mark your answers on the multiple choice sheet provided. **Marks**

1. The region in the first quadrant between the x axis and $y = 6x - x^2$ is rotated about the y axis. The volume of this solid of revolution is **1**

(A) $\displaystyle\int_0^6 \pi \left(6x - x^2\right)\, dx$

(B) $\displaystyle\int_0^6 2\pi x \left(6x - x^2\right)\, dx$

(C) $\displaystyle\int_0^6 \pi x \left(6x - x^2\right)^2\, dx$

(D) $\displaystyle\int_0^6 \pi \left(3 + \sqrt{9 - y}\right)^2\, dx$

2. What are all the values of k for which the graph of $y = x^3 - 3x^2 + k$ will have three distinct x intercepts? **1**

(A) all $k > 0$

(B) all $k < 4$

(C) $k = 0, 4$

(D) $0 < k < 4$

3. Which of the following is the triple root of the equation **1**

$$8x^4 + 12x^3 - 30x^2 + 17x - 3 = 0$$

(A) $\frac{1}{2}$ (B) $-\frac{5}{4}$ (C) -3 (D) 0

4. If n is a non-negative integer, then for what values of n is $\displaystyle\int_0^1 x^n\, dx = \int_0^1 (1-x)^n\, dx$ true? **1**

(A) no solution

(B) n even, only

(C) non zero n, only

(D) all values of n

5. What are the coordinates of the foci of $xy = 18$? **1**

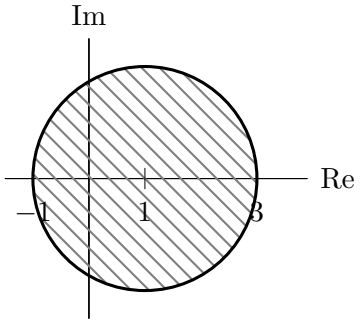
(A) $(0, 6)$, $(0, -6)$

(B) $\left(0, 3\sqrt{2}\right)$, $\left(0, -3\sqrt{2}\right)$

(C) $\left(3\sqrt{2}, 3\sqrt{2}\right)$, $\left(-3\sqrt{2}, -3\sqrt{2}\right)$

(D) $(6, 6)$, $(-6, 6)$

6. Which of the following inequalities is represented by the Argand diagram? **1**



(A) $|z-1| \leq 2$

(B) $|z-i| \leq 2$

(C) $|z+1| \leq 2$

(D) $|z+i| \leq 2$

7. What does $\displaystyle\int \frac{dx}{(x-1)(x+2)}$ evaluate to? **1**

(A) $\frac{1}{3}\log_e\left|\frac{x-1}{x+2}\right| + C$

(B) $\frac{1}{3}\log_e\left|\frac{x+2}{x-1}\right| + C$

(C) $\frac{1}{3}\log_e|(x-1)(x+2)| + C$

(D) $(\log_e|x-1|)(\log_e|x+2|)$

8. What is the value of $\displaystyle\int_0^1 xe^{-x}\,dx$? **1**

(A) $1-2e$

(B) -1

(C) $1-2e^{-1}$

(D) $2e-1$

9. What is the value of the eccentricity of the hyperbola $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$? **1**

(A) $\dfrac{3}{\sqrt{13}}$

(B) $\dfrac{\sqrt{13}}{3}$

(C) $\dfrac{11}{\sqrt{5}}$

(D) $\sqrt{2}$

10. What is the value of $\dfrac{dy}{dx}$ at the point $(1,2)$ if $xy^2 + 2xy = 8$? **1**

(A) $-\dfrac{5}{2}$

(B) $-\dfrac{4}{3}$

(C) -1

(D) $-\dfrac{1}{2}$

End of Section I
Examination continues overleaf...

Section II: Short answer

Question 11 (15 Marks) Commence a NEW page. **Marks**

(a) Evaluate:

i. $\displaystyle\int \frac{dx}{\sqrt{7-9x-x^2}}$ **2**

ii. $\displaystyle\int \frac{dx}{x\log_e x}$ **2**

(b) Evaluate $\displaystyle\int_1^2 \frac{dx}{x\,(1+x^2)}$. **4**

(c) Evaluate $\displaystyle\int \frac{x}{\sqrt{1-x}}\,dx$. **3**

(d) Find $\displaystyle\int e^{-2x}\cos x\,dx$. **4**

Question 12 (15 Marks) Commence a NEW page. **Marks**

(a) Show that $3i$ is a root of $P(x) = x^4 - 3x^3 + 5x^2 - 27x - 36$, and hence solve $P(x) = 0$ completely. **3**

(b) If α, β and γ are roots of $3x^3 + 4x^2 + 5x + 1 = 0$, find the value of **3**

$$\frac{1}{\alpha^2\beta^2} + \frac{1}{\beta^2\gamma^2} + \frac{1}{\alpha^2\gamma^2}$$

(c) Given $Q(x) = x^4 - 5x^3 + 4x^2 + 3x + 9$ has a root of multiplicity 2, solve $Q(x) = 0$ over $\mathbb{C}$. **3**

(d) The roots of the polynomial equation $x^3 - 2x^2 + 3x + 1 = 0$ are α, β and γ. **3**

Find the value of $\alpha^3 + \beta^3 + \gamma^3$.

(e) The polynomial $x^5 - ax^2 + b = 0$ has a multiple root. **3**

Show that $108a^5 = 3\,125b^3$.

Question 13 (15 Marks) Commence a NEW page. **Marks**

(a) Sketch the region in the Argand diagram which simultaneously satisfies the following inequalities: **2**

$$\begin{cases} |z-2i| \leq 2 \\ \text{Im}(z) \geq 2 \end{cases}$$

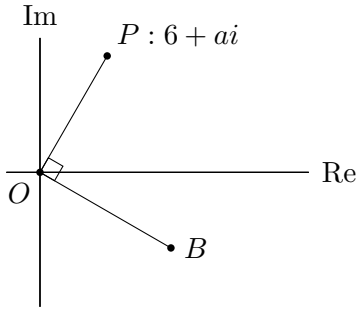
(b) What is the locus in the Argand diagram of the point z such that **3**

$$z\overline{z} - 2(z+\overline{z}) = 5$$

(c) Find the value of z^{10} in Cartesian form, given that **3**

$$z = \sqrt{2} - \sqrt{2}i$$

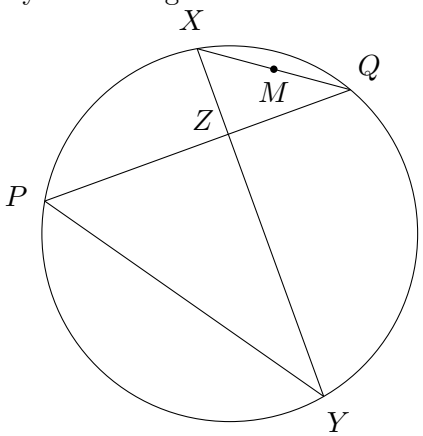
(d) In the following Argand diagram, P represents the point $6+ai$, and O is the origin. **3**



Find the complex number represented by the point B, given $\angle POB = 90^\circ$ and

$$2\,|OB| = 3\,|OP|$$

(e) Two perpendicular chords PQ and XY of a circle intersect at Z. **4**
Copy the diagram into your writing booklet.



If M is the midpoint of the chord QX, prove that MZ produced is perpendicular to the chord PY.

Question 14 (15 Marks) Commence a NEW page. **Marks**

(a) Sketch the following graphs:

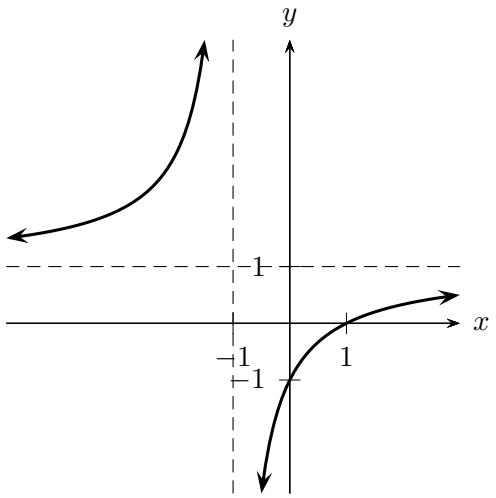
i. $y = |\sin x|$ for $-2\pi \leq x \leq 2\pi$. **1**

ii. $y = \sqrt{x^2 - 4}$ **2**

iii. $y^2 = x^2 - 9x$ **2**

(b) Sketch $y = \dfrac{1}{(x-1)^2(x+2)}$. **2**

(c) The diagram shows the graph of $f(x)$.



Sketch the following curves on separate diagrams, clearly indicating any turning points and asymptotes.

i. $y = \dfrac{1}{f(x)}$ **2**

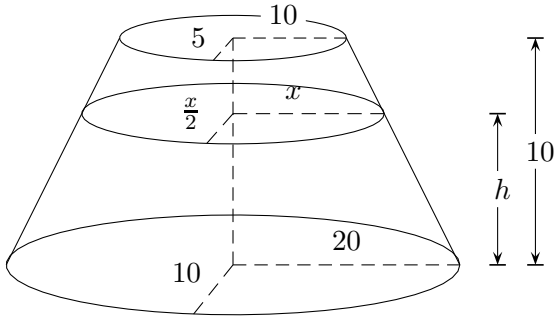
ii. $y = f(|x|)$ **2**

iii. $y = \log_e(f(x))$ **2**

iv. $y = e^{f(x)}$ **2**

Question 15 (15 Marks) Commence a NEW page. **Marks**

(a) A solid of height 10 m stands on horizontal ground. The base of the solid is an ellipse with semi-axes of 20 m and 10 m. Horizontal cross-sections taken parallel to the base and at height h metres above the base are ellipses with semi-axes x metres and $\frac{x}{2}$ metres so that the centres of these elliptical cross-sections lie on a vertical straight line, and the extremitites of their semi-axes line on sloping straight lines as shown in the diagram. The top of the solid is an ellipse with semi-axes 10 m and 5 m. **5**

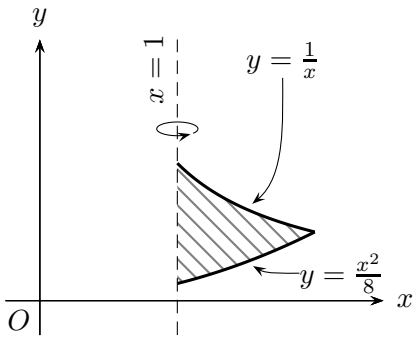


Show that the volume V m^3 of the solid is given by

$$V = \frac{\pi}{2}\int_0^{10} (20-h)^2\, dh$$

and hence find the volume correct to the nearest cubic metre.

(b) The shaded region shown in the diagram below is bounded by $y = \dfrac{1}{x}$, $y = \dfrac{x^2}{8}$ and $x = 1$. This region is rotated about the line $x = 1$.



i. Find an integral which gives the volume of the resulting solid of revolution using the method of cylindrical shells. **4**

ii. Find the volume of the solid of revolution. **2**

(c) On the number plane, shade the region **4**

$$(x-a)^2 + (y-b)^2 \leq R^2$$

where $R < b < a$.

Find the volume when this shape is rotated about the y axis using the method of slices.

Question 16 (15 Marks) Commence a NEW page. **Marks**

(a) i. Determine the real values of p for which the equation

$$\frac{x^2}{3+p} + \frac{y^2}{8+p} = 1$$

defines

(α) an ellipse **1**

(β) a hyperbola **2**

ii. For the value $p = -4$ in the above equation, find the **2**

- eccentricity
- coordinates of the foci, and
- the equations of the directrices

of the conic.

(b) P is a point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with centre at the origin O.

A line drawn from the origin O, parallel to the tangent to the ellipse at P, meets the ellipse at Q.

i. Derive the equation of the tangent at $P\,(a\cos\theta, b\sin\theta)$. **2**

ii. Hence or otherwise, prove that the area of $\triangle OPQ$ is independent of the position of P. **3**

(c) i. Find the equation of the normal at $P(a\sec\theta, b\tan\theta)$ to the hyperbola **2**

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

ii. This normal intersects the x and y axes at Q and R respectively. $M(x, y)$ is the midpoint of QR. Find the equation of the locus of M as P varies on the hyperbola. **3**

End of paper.

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, \qquad n \neq -1; \quad x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x + C, \qquad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C, \qquad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax + C, \qquad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax + C, \qquad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax + C, \qquad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax + C, \qquad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C, \qquad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a} + C, \qquad a > 0, -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln \left(x + \sqrt{x^2 - a^2} \right) + C, \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln \left(x + \sqrt{x^2 + a^2} \right) + C$$

NOTE: $\ln x = \log_e x$, $x > 0$

Answer sheet for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

STUDENT NUMBER: .....

Class (please ✔)

◯ 12M4A – Mr Weiss ◯ 12M4B – Mr Ireland ◯ 12M4C – Mr Fletcher

1 – Ⓐ Ⓑ Ⓒ Ⓓ
2 – Ⓐ Ⓑ Ⓒ Ⓓ
3 – Ⓐ Ⓑ Ⓒ Ⓓ
4 – Ⓐ Ⓑ Ⓒ Ⓓ
5 – Ⓐ Ⓑ Ⓒ Ⓓ
6 – Ⓐ Ⓑ Ⓒ Ⓓ
7 – Ⓐ Ⓑ Ⓒ Ⓓ
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