Ext 2

NORTH SYDNEY BOYS HIGH SCHOOL

2009
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Examiner: B. Weiss

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write on one side of the paper (with lines) in the booklet provided
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.**

- Attempt all questions

Class Teacher:
(Please tick or highlight)
- ☑ Mr Barrett
- O Mr Fletcher
- O Mr Weiss

Ben L

Student Number 194 88152

(To be used by the exam markers only.)

Question No	1	2	3	4	5	6	7	8	Total	Total
Mark	11/15	14/15	13/15	14/15	12/15	15/15	8 +3 /15	15/15	102/120	/100

H.

Question 1

(a) Find the following integrals:

(i) $\int \tan^3 x \, dx$ 3

(ii) $\int \frac{dx}{x^2 - 6x + 13}$ 2

(b) Evaluate

(i) $\int_0^1 \frac{x}{\sqrt{4 - x^2}} \, dx$ 3

(ii) $\int_0^{\frac{\pi}{2}} x \cos^2 x \, dx$ 3

(c) (i) Show that if $I_n = \int_0^1 x^n e^{-x} \, dx$, then $I_n = n \cdot I_{n-1} - \frac{1}{e}$ 2

(ii) Hence find $\int_0^1 x^3 e^{-x} \, dx$. 2

Question 2 (Start a new page)

(a) Find $\sqrt{6i - 8}$, and hence solve the equation $2z^2 - (3 + i)z + 2 = 0$. 4

(b) Solve $3x^3 - 10x^2 + 7x + 10 = 0$ given that $x = 2 - i$ is a root of the equation. 3

(c) The polynomial equation $P(x) = x^3 + px^2 + q = 0$ has roots α, β and γ. Form the polynomial equation with roots given by

(i) $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$ 2

(ii) α^2, β^2 and γ^2 2

(d) Use the method of cylindrical shells to find the voume of the solid generated by rotating the region bounded by $y = \ln x$, the x-axis and the lines $x = 1$ and $x = e$, about the y-axis. 4

Question 3 (Start a new page)

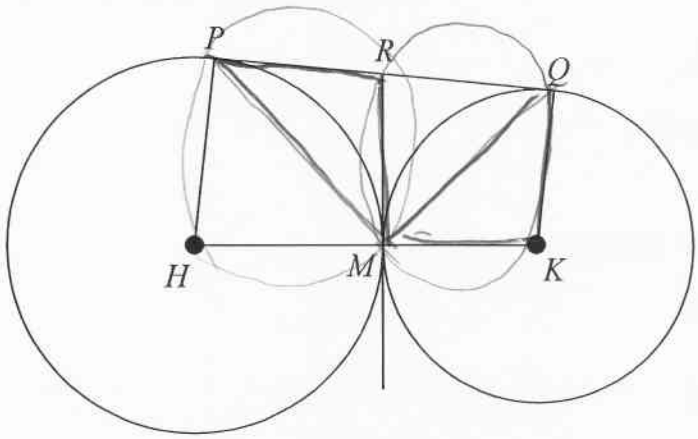
(a) (i) Prove that the equation of the tangent at the point $\left(t, \frac{1}{t}\right)$ to the hyperbola $xy = 1$ is $x + t^2y = 2t$. **2**

(ii) The tangent at a point P on the hyperbola $xy = 1$ meets the y-axis at A, and the normal at P meets the x-axis at B. Find the equation of the locus of the midpoint of AB as P moves on the hyperbola. (Draw a diagram) **3**

(b) $P(a\cos\alpha,\ b\sin\alpha)$ and $Q(a\cos\beta,\ b\sin\beta)$ are the endpoints of a focal chord of the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. **3**

Show that $e = \dfrac{\sin(\alpha - \beta)}{\sin\alpha - \sin\beta}$.

(c) Shown below are two circles with centres H and K which touch at M. PQ and RM are common tangents.



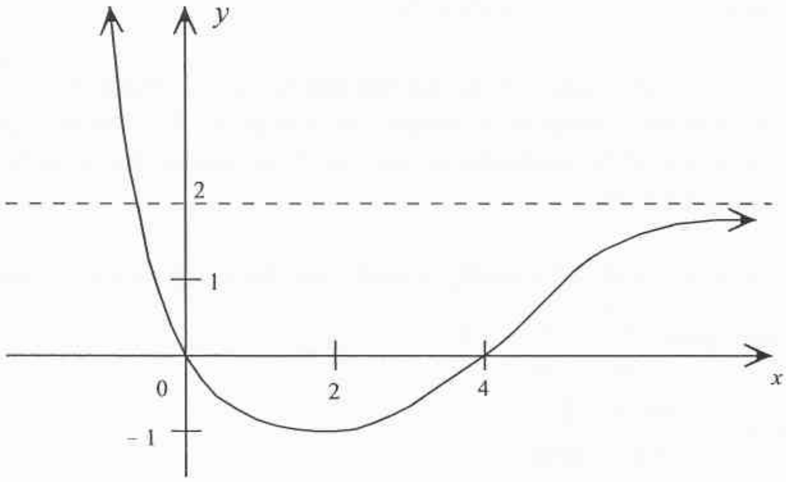
(i) Show that quadrilaterals $HPRM$ and $MRQK$ are cyclic. **2**

(ii) Prove tha triangles PRM and MKQ are similar. **2**

(d) Show that the polynomial equation $4x^3 + 20x^2 - 23x + 6 = 0$ has a double root, and find the value of each of its roots. **3**

Question 4

(a) The diagram shows the graph of $y = f(x)$.



Sketch on separate diagrams, the following curves, indicating clearly any turning points and asymptotes.

(i) $y = \dfrac{1}{f(x)}$ **1**

(ii) $y = [f(x)]^2$ **2**

Draw neat sketches of the following:

(b) $y = x \sin x$ **2**

(c) $y = \sin^{-1}(\sin x)$ **2**

(d) $y = x^2 - \dfrac{1}{x}$ **3**

(e) (i) $f(x) = \dfrac{x^2 - 4}{x - 3}$ **3**

(ii) $[f(x)]^2 = \dfrac{x^2 - 4}{x - 3}$ **2**

Question 5 (Start a new page)

(a) (i) Express the complex number $z = -\sqrt{3} + i$ in mod-arg form. **1**

(ii) Hence, or otherwise, show that $z^7 + 64z = 0$. **2**

(b) Find the equation, in Cartesian form, of the locus of the point z if

$$\text{Re}\left[\frac{z-4}{z}\right] = 0$$ **3**

(c) Sketch the region S in the complex plane, where

$$S = \left\{ |z| \leq 1 \quad \text{and} \quad 0 \leq \arg z < \frac{\pi}{3} \right\}$$ **2**

(d) (i) Use de Moivre's theorem to express $\cos 5\theta$ and $\sin 5\theta$ in terms of $\sin\theta$ and $\cos\theta$. **3**

(ii) Hence express $\tan 5\theta$ as a rational function of t, where $t = \tan\theta$. **2**

(iii) Find $\tan\frac{\pi}{5} \cdot \tan\frac{2\pi}{5} \cdot \tan\frac{3\pi}{5} \cdot \tan\frac{4\pi}{5}$ **2**

Question 6 (Start a new page)

(a) A particle of mass 1 kg is projected upwards with initial speed $10\ \text{ms}^{-1}$. **5**
The air resistance is given by $R = \frac{1}{10}V^2$.
Take the acceleration due to gravity to be $10\ \text{ms}^{-2}$.
Find the maximum height reached, and the time taken to reach this height.

(b) Find the largest coefficient in the expansion of $(2x+3)^{21}$. **3**

(c) If $x^m y^n = k$, where k is a constant, show that $\frac{dy}{dx} = -\frac{my}{nx}$. **3**

(d) Use the expansion of $(1+x)^{2n}$ to show that

(i) $\binom{2n}{1} + \binom{2n}{2} + \binom{2n}{3} + \ldots + \binom{2n}{2n} = 4^n - 1$ **2**

(ii) Use the identity $(1+x)^{2n} \equiv (1+x)^n (1+x)^n$ to show that **2**

$$\binom{2n}{2} = 2\cdot\binom{n}{2} + \binom{n}{1}^2$$

Question 7 (Start a new page)

(a) Use the process of mathematical induction to show that

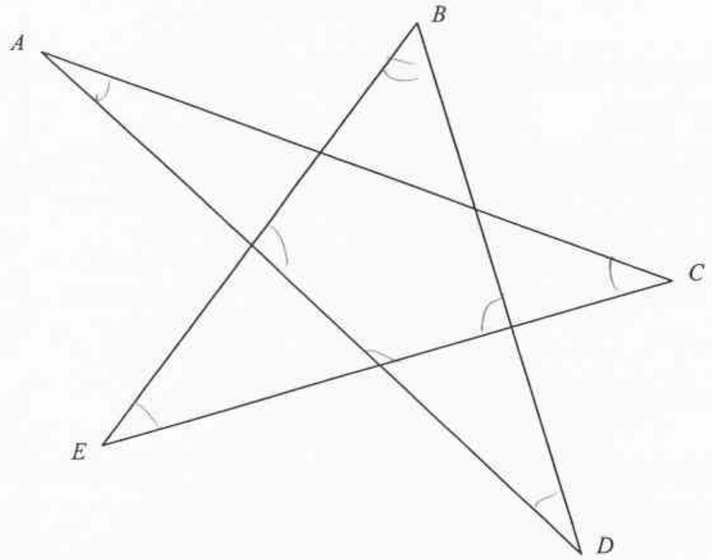
$$\sum_{k=1}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!}$$ **5**

(b) With the aid of a diagram, show that the area enclosed by the ellipse **4**

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is given by $A = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$.

Hence show that the area of the ellipse is πab.

(c) **2**



Prove that $\angle A + \angle B + \angle C + \angle D + \angle E = 180°$

(d) The base of a solid is the region bounded by the parabolas $x = y^2$ and $x = 4 - 3y^2$, and the cross-sections perpendicular to the x-axis are squares.

(i) Draw a neat sketch of this solid. **1**

(ii) Find the volume of the solid. **3**

$3y^2 = 4 - x$

$y^2 = \frac{4-x}{3}$

$y = \sqrt{\frac{4-x}{3}}$

Question 8 (Start a new page)

(a) If a and b are positive numbers such that $a + b = 1$, prove that

(i) $a + b \geq 2\sqrt{ab}$ 1

(ii) $\frac{1}{a} + \frac{1}{b} \geq 4$ 2

(iii) $a^2 + b^2 \geq \frac{1}{2}$ 2

(iv) $\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right) \geq 9$ 2

(b) A particle is projected from ground level so that it just clears two poles of height h at distances of b and c metres from the point of projection. If v m/s is the velocity of projection, and θ is the angle of projection to the horizontal:

(i) Show that $y = x\tan\theta - \frac{gx^2}{2v^2} \cdot \sec^2\theta$ 2

(ii) Show that $v^2 = \frac{(b+c)g\sec^2\theta}{2\tan\theta}$ 3

(iii) Hence or otherwise show that $\tan\theta = \frac{h(b+c)}{bc}$ 3

BLANK PAGE