Ext 2

NORTH SYDNEY BOYS HIGH SCHOOL

2007
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write on one side of the paper (with lines) in the booklet provided
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.**

- Attempt all questions

Class Teacher:
(Please tick or highlight)
- O Mr Ee
- O Mr Trenwith
- O Mr Weiss

Student Number:________________

(To be used by the exam markers only.)

Question No	1	2	3	4	5	6	7	8	Total	Total
Mark	/15	/15	/15	/15	/15	/15	/15	/15	/120	/100

Mark

QUESTION 1 (15 marks)

(a) Find $\int \frac{x}{\sqrt{16-x^2}}\,dx$ 2

(b) By completing the square, find $\int \frac{8}{x^2+4x+13}\,dx$ 2

(c) Use integration by parts to evaluate $\int_1^e x^4 \log_e x\,dx$. 4

(d) Use the substitution $u = \cos x$ to find $\int \cos^2 x \sin^5 x\,dx$ 3

(e) Express $\frac{3x+7}{(x+1)(x+2)(x+3)}$ in partial fractions and hence prove that

$$\int_0^1 \frac{3x+7}{(x+1)(x+2)(x+3)}\,dx = \ln 2$$ 4

QUESTION 2 (15 marks) Start a new page

(a) Let $z = 2+i$ and $w = 1-i$. Find, in the form $x+iy$,

(i) $3z + iw$ 1

(ii) $z\bar{w}$ 1

(iii) $\frac{5}{z}$ 1

(b) Let $\alpha = -\sqrt{3}+i$.

(i) Express α in modulus-argument form. 2

(ii) Express α^4 in modulus-argument form. 2

(iii) Hence express α^4 in the form $x+iy$. 1

QUESTION 2 (Continued)

(c) If $z_1 = 4+i$ and $z_2 = 1+2i$ show geometrically how to construct the vectors representing.

(i) $z_1 + z_2$. **1**

(ii) $z_1 - z_2$. **1**

(d) Consider the hyperbola with the equation $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

(i) Find the coordinates of the foci and x-intercepts of the hyperbola. **2**

(ii) Find the equations of the directrices and the asymptotes of the hyperbola. **2**

(iii) What are the parametric equations of this hyperbola? **1**

QUESTION 3 (15 marks) Start a new page

(a) If α, β and γ are the roots of the cubic equation $x^3 + mx + n = 0$, find in terms of m and n, the values of

(i) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ **3**

(ii) $\alpha^3 + \beta^3 + \gamma^3$ **2**

(iii) Determine the cubic equation whose roots are α^2, β^2 and γ^2. **3**

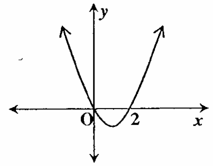
(b) Given that the equation $x^4 - 5x^3 - 9x^2 + 81x - 108 = 0$ has a triple root, find all the roots of the equation. **4**

(c) If $y = e^{-x}(A\sin 2x + B\cos 2x)$, prove that **3**

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$$

QUESTION 4 (15 marks) Start a new page

(a) Given $f(x) = x^2 - 2x$. On separate diagrams sketch the graphs of the following. Indicate clearly any asymptotes, intercepts with the axes and local maxima and minima.



(i) $y = |f(x)|$ **1**

(ii) $y = f(|x|)$ **1**

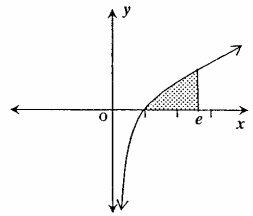
(iii) $y = \frac{1}{f(x)}$ **2**

(iv) $y^2 = f(x)$ **2**

(v) $y = [f(x)]^2$ **1**

(vi) $y = \ln[f(x)]$ **2**

(b) The region bounded by $y = \ln x$, $x = e$ and the x – axis is rotated about the y – axis. Find the volume of rotation. Use the method of cylindrical shells. **3**



(b) First differentiating both sides of the formula

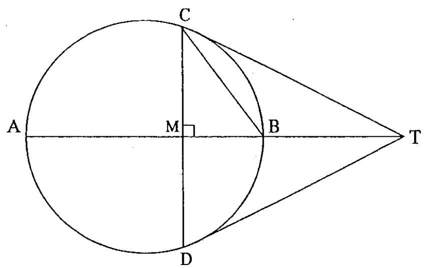
$$1 + x + x^2 + x^3 + \ldots + x^n = \frac{x^{n+1} - 1}{x - 1}$$

then find an expression for

$1 + 2\times 2 + 3\times 4 + 4\times 8 + \ldots + n2^{n-1}$ **3**

QUESTION 5 (15 marks) Start a new page

(a) In the circle shown below, the diameter AB meets the chord CD at right angles at M. The tangents at C and D meet at T.



(i) Show that BC bisects $\angle MCT$. 2

(ii) Show that triangle BCM is similar to triangle CAM. Hence, show that $CM^2 = AM \times BM$ 2

(iii) Show that $TB \times TA = MB \times TA + TB \times TM$ 4

(iv) Hence, or otherwise, show that M divides the interval AB internally in the same ratio that T divides AB externally. 2

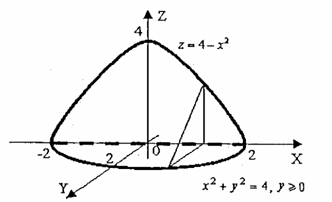
(c) If $I_n = \int \tan^n x \, dx$

(i) Show that $I_n = \dfrac{1}{n-1}\tan^{n-1} x - I_{n-2}$ 3

(ii) Find $\int \tan^6 x \, dx$. 2

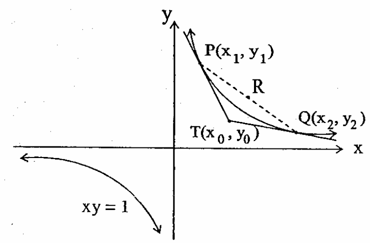
QUESTION 6 (15 marks) Start a new page

(a) The solid shown has a semicircular base of radius 2 units. Vertical cross-sections perpendicular to the diameter are right-angled triangles whose height is bounded by the parabola $z = 4 - x^2$.



By slicing at right angles to the x- axis , show that the volume of the solid is given by $V = \int_0^2 (4 - x^2)^{3/2} dx$, and hence calculate this volume. 5

(b) The tangents at $P(x_1, y_1)$, $Q(x_2, y_2)$ on the hyperbola $xy = 1$ intersect at the point $T(x_0, y_0)$.



(i) Show that the tangent at $P(x_1, y_1)$ has equation $xy_1 + yx_1 = 2$ 2

(ii) Show that the chord of contact PQ has equation $xy_0 + yx_0 = 2$ 2

(iii) Show that x_1 and x_2 are the roots of the quadratic equation $y_0 x^2 - 2x + x_0 = 0$ 2

(iv) Hence, or otherwise, show that the midpoint R, of PQ has coordinates $\left(\dfrac{1}{y_0}, \dfrac{1}{x_0}\right)$ 2

(v) Hence, or otherwise, show that as T moves on the hyperbola $xy = c^2$, $0 < c < 1$, R moves on the hyperbola $xy = \dfrac{1}{c^2}$ 2

QUESTION 7 (15 marks) Start a new page

(a) (i) Show that $\tan(A+\frac{\pi}{2})=-\cot A$ 1

(ii) Use mathematical induction to prove that 4

$$\tan\left[(2n+1)\frac{\pi}{4}\right]=(-1)^n \qquad \text{for all integer } n \geq 1.$$

(b) Two stones are thrown simultaneously form the same point in the same direction with the same non-zero angle of projection (upward inclination to the horizontal), α, but with different velocities U, V metres per second (U < V).

The slower stone hits the ground at a point P on the same level as the point of projection. At that instant the faster stone just clears a wall of height h metres above the level of projection and its (downward) path makes an angle β with the horizontal.

(i) Show that while the stones are in flight, the line joining them has a gradient of $\tan\alpha$. 3

(ii) Hence, express the horizontal distance from P to the foot of the wall in terms of h and α. 2

(iii) Show that $V(\tan\alpha+\tan\beta)=2U\tan\alpha$. 3

(iv) Hence, deduce that, if $\beta=\frac{1}{2}\alpha$, then $U<\frac{3}{4}V$. 2

QUESTION 8 (15 marks) Start a new page

(a) Show that the locus in the Argand plane represented by the equation $|z-1|+|z+1|=4$ is a conic and find its cartesian equation. 3

(b) A particle of mass m is projecteds against a constant gravitational force mg and resistance $\frac{mv}{k}$, where v is the velocity of the particle and k is a constant.
Let x be the distance traveled in time t.
Initially the particle has zero displacement and $v_o=k(h-g)$, where h is a constant.

(i) Show that the equation of motion of the particle is $\ddot{x}=-\left[\frac{kg+v}{k}\right]$ 1

(ii) Show that $t=k\log\left(\frac{kh}{kg+v}\right)$ 2

(iii) Find the time taken by the particle to reach the maximum height, H, and determine the height of that point. 3

(c) A polynomial P(x) is divided by x^2-a^2 where $a\neq 0$,and the remainder is $px+q$.

(i) Show that $p=\frac{1}{2a}[P(a)-P(-a)]$

and $q=\frac{1}{2}[P(a)+P(-a)]$ 3

(ii) Find the remainder when $P(x)=x^n-a^n$, for n a positive integer, is divided by x^2-a^2 . 3

2007 Trial HSC Ext 2 Suggested Solutions

Q1(a) Let $u = 16 - x^2$

$\frac{du}{dx} = -2x \qquad dx = -\frac{du}{2x}$

$\int \frac{x}{\sqrt{16-x^2}}\,dx = \int \frac{x}{\sqrt{u}} \cdot \frac{du}{-2x}$

$= -\frac{1}{2}\int u^{-\frac{1}{2}}\,du$

$= -\frac{1}{2}(2\sqrt{u}) + c$

$= -\sqrt{16-x^2} + c$

(b) $\int \frac{8}{x^2+4x+13}\,dx$

$= \int \frac{8}{(x+2)^2+9}\,dx$

$= \frac{8}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + c$

(c) $\int_1^e x^4 \ln x\,dx$

$= \left[\frac{x^5}{5}\ln x\right]_1^e - \int_1^e \frac{x^5}{5}\cdot\frac{1}{x}\,dx$

$= \frac{e^5}{5} - \frac{1}{5}\left[\frac{x^5}{5}\right]_1^e$

$= \frac{e^5}{5} - \frac{e^5}{25} + \frac{1}{25} = \frac{4e^5+1}{25}$

d) Let $u = \cos x$

$\frac{du}{dx} = -\sin x \qquad du = -\sin x\,dx$

$\int \cos^2 x \sin^5 x\,dx$

$= \int \cos^2 x \sin^4 x (\sin x\,dx)$

$= \int u^2(1-u^2)^2 - du$

$= -\int u^2 - 2u^4 + u^6\,du$

$= -\frac{1}{3}u^3 + \frac{2}{5}u^5 - \frac{1}{7}u^7 + c$

$= -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + c$

(e) $\frac{3x+7}{(x+1)(x+2)(x+3)} \equiv \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$

$3x+7 \equiv A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$

Put $x=-1 \quad 4 = 2A \Rightarrow A = 2$

$x=-2, \quad 1 = -B \Rightarrow B = -1$

$x=-3, \quad -2 = 2C \Rightarrow C = -1$

$\int_0^1 \frac{3x+7}{(x+1)(x+2)(x+3)}\,dx$

$= \int_0^1 \frac{2}{x+1} - \frac{1}{x+2} - \frac{1}{x+3}\,dx$

$= \left[2\ln(x+1) - \ln(x+2) - \ln(x+3)\right]_0^1$

$= (2\ln 2 - \ln 3 - \ln 4) - (2\ln 1 - \ln 2 - \ln 3)$

$= \ln\frac{4}{12} + \ln 6 = \ln\frac{24}{12} = \ln 2$

Q2.(a) $z = 2+i, \quad w = 1-i$

(i) $3z + iw$

$= 3(2+i) + i(1-i)$

$= 6 + 3i + i + 1$

$= 7 + 4i$ (1)

(ii) $z\bar{w}$

$= (2+i)(1+i)$

$= 2 + i + 2i - 1$

$= 1 + 3i$ (1)

(iii) $\frac{5}{z}$

$= \frac{5}{2+i} \times \frac{2-i}{2-i}$

$= \frac{10-5i}{4+1} = 2 - i$ (1)

(b) $\alpha = -\sqrt{3} + i$

(i) $|\alpha| = (-\sqrt{3})^2 + 1^2$

$= \sqrt{4} = 2$ (1)

$\arg\alpha = \tan^{-1}\frac{1}{-\sqrt{3}}$

$= \frac{5\pi}{6}$ (1)

$\alpha = 2\,\text{cis}\left(\frac{5\pi}{6}\right)$

(ii) $\alpha^4 = 2^4\,\text{cis}\left(4\times\frac{5\pi}{6}\right)$

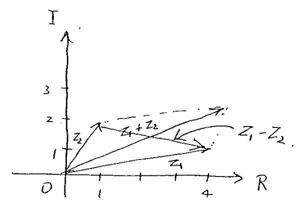
$= 16\,\text{cis}\left(-\frac{2\pi}{3}\right)$ (2)

(iii) $\alpha^4 = 16\left(\cos -\frac{2\pi}{3} + i\sin -\frac{2\pi}{3}\right)$

$= 16\times -\frac{1}{2} + i\left(16\times -\frac{\sqrt{3}}{2}\right)$

$= -8 - 8\sqrt{3}\,i$ (1)

(c)



(d)(i) $\frac{x^2}{16} - \frac{y^2}{9} = 1$

$a = 4, \quad b = 3$

$b^2 = a^2(e^2-1)$

$9 = 16(e^2-1)$

$e^2 - 1 = \frac{9}{16}$

$e^2 = \frac{9}{16} + 1 = \frac{25}{16}$

$e = \frac{5}{4}$

$S \equiv (ae, 0), \quad S' \equiv (-ae, 0)$

$\equiv (5, 0) \qquad \equiv (-5, 0)$ (1)

when $y = 0, \quad \frac{x^2}{16} = 1$

$x = \pm 4$ (1)

$\therefore$ x-intercepts are 4 & -4.

(ii) Directrices are

$x = \pm\frac{a}{e}$

$= \pm\frac{16}{5}$ (1)

Eq of asymptotes are

$y = \pm\frac{b}{a}x$

$= \pm\frac{3}{4}x$ (1)

(iii) $x = 4\sec\theta$

$y = 3\tan\theta$ (1)

(a) $x^3 + mx + n = 0$.

$\alpha\beta\gamma = -n$

$\alpha + \beta + \gamma = 0$

$\alpha\beta + \alpha\gamma + \beta\gamma = m$.

(i) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

$= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} = \frac{m}{-n}$

(ii) $\alpha^3 + m\alpha + n = 0$ — (1)

$\beta^3 + m\beta + n = 0$ — (2)

$\gamma^3 + m\gamma + n = 0$ — (3)

(1)+(2)+(3)

$\alpha^3 + \beta^3 + \gamma^3 + m(\alpha + \beta + \gamma) + 3n = 0$.

$\alpha^3 + \beta^3 + \gamma^3 = -3n$.

(iii) $(\sqrt{x})^3 + m\sqrt{x} + n = 0$

$\sqrt{x}(x + m) + n = 0$

$\sqrt{x} = \frac{-n}{x+m}$

$x = \frac{n^2}{(x+m)^2}$

$x(x+m)^2 - n^2 = 0$.

$x^3 + 2mx^2 + m^2x - n^2 = 0$.

(b) $P(x) = x^4 - 5x^3 - 9x^2 + 81x - 108$

$P'(x) = 4x^3 - 15x^2 - 18x + 81$

$P''(x) = 12x^2 - 30x - 18$. (1)

When $P''(x) = 0$ $12x^2 - 30x - 18 = 0$.

$x = -\frac{1}{2}$ or $x = 3$ (1)

$P'(3) = P(3) = 0$.

$\therefore$ $x = 3$ is a triple root of $P(x) = 0$ (1)

If the other root is α

Sum of roots $= 9 + \alpha = 5$.

$\alpha = -4$. (1)

$\therefore$ Roots of $P(x) = 0$ are $3, 3, 3, + -4$.

(c) $y = e^{-x}(A\sin 2x + B\cos 2x)$

$e^x y = A\sin 2x + B\cos 2x$.

$e^x\frac{dy}{dx} + ye^x = 2A\cos 2x - 2B\sin 2x$ (1)

$e^x\frac{d^2y}{dx^2} + \frac{dy}{dx}e^x + ye^x + e^x\frac{dy}{dx}$

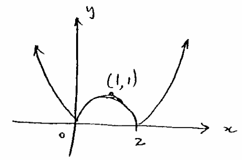
$= -4A\sin 2x - 4B\cos 2x$ (1)

$= -4e^x y$.

$e^x\left(\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y\right) = 0$.

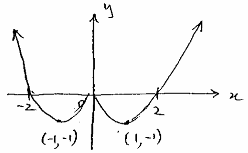
$\therefore \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$. (1)

Q4. (a) (i) $y = |f(x)|$



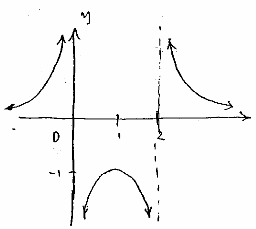
(1)

(ii) $y = f(|x|)$.



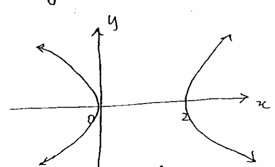
(1)

(iii)



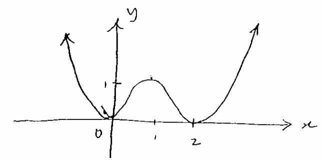
(2)

(iv) $y^2 = f(x)$.



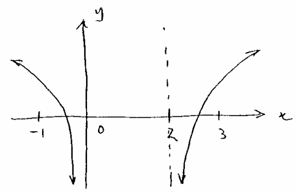
(2)

(v) $y = [f(x)]^2$

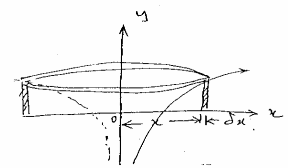


(1)

(vi) $y = \ln[f(x)]$.



(b).



$\delta V = [\pi(x + \delta x)^2 - \pi x^2]y$

$= 2\pi xy\delta x + \pi y\delta x^2$

$\doteqdot 2\pi xy\delta x$.

$V = \lim_{\delta x \to 0} \sum_{x=1}^{e} 2\pi xy\delta x$

$= \int_1^e 2\pi xy\,dx$ (1)

$= 2\pi\int_1^e x\ln x\,dx$

$= 2\pi\left[\frac{x^2}{2}\ln x\right]_1^e - 2\pi\int_1^e \frac{x}{2}dx$

$= \pi\left[x^2\ln x - \frac{x^2}{2}\right]_1^e$

$= \pi\left[(e^2 - \frac{e^2}{2}) - (0 - \frac{1}{2})\right]$

$= \frac{\pi}{2}(e^2 + 1)$ unit^3. (2)

Q4(c) $1 + x + x^2 + \cdots + x^n = \dfrac{x^{n+1} - 1}{x - 1}$.

Diff. both sides w.r.t. x

$1 + 2x + 3x^2 + \cdots + nx^{n-1} = \dfrac{(n+1)x^n(x-1) - (x^{n+1}-1)}{(x-1)^2}$

$= \dfrac{(n+1)x^{n+1} - (n+1)x^n - x^{n+1} + 1}{(x-1)^2}$

$= \dfrac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}$.

Put $x = 2$.

$1 + 2\times 2 + 3\times 4 + 4\times 8 + \cdots + n2^{n-1}$

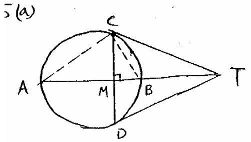
$= \dfrac{n\times 2^{n+1} - (n+1)2^n + 1}{1}$

$= n\times 2^{n+1} - 2^n(n+1) + 1$

$= 2^n\left[2n - (n+1)\right] + 1$

$= 2^n(n+1) + 1$.

Q5(a)



Join AC + BC

$\angle ACB = 90^\circ$ ($\angle$ in a semi-$\odot$)

$\angle TCB = \angle BAC$ ($\angle$ bet. tangent + chord $= \angle$ in alt. seg).

$= \alpha$

$\therefore \angle ACM = 90^\circ - \alpha$ ($\angle$ sum of $\triangle$)

But $\angle ACB = 90^\circ$

$\therefore \angle BCM = 90^\circ - (90^\circ - \alpha) = \alpha$.

$\therefore \angle TCB = \angle BCM$

i.e. BC bisects $\angle TCM$. (2)

(ii) In $\triangle BCM$ + $\triangle CAM$.

$\angle BMC = \angle AMC = 90^\circ$ ($CM \perp AB$)

$\angle CAM = \angle BCM = \alpha$. (proved above)

$\therefore \triangle BCM \,|||\, \triangle CAM$ (equiangular) (1)

$\dfrac{CM}{AM} = \dfrac{BM}{CM}$ (corresp. sides of similar $\triangle$'s)

$\therefore CM^2 = AM \times BM$ (1)

(iii) $CT^2 = TA \times TB$

(The sq. on the tangent equal the product of the segs. of any chord through the pt.) (1)

In $\triangle TCM$

$CT^2 = CM^2 + MT^2$ (1)

$= AM \times BM + MT^2$

$TA \times TB = CT^2$

$= AM \times BM + MT^2$

$= (TA - TM) \times MB + MT^2$ (1)

$= TA \times MB - TM \times MB + MT^2$

$= TA \times MB - TM(TM - MB)$

$= TA \times MB - TM \times TB$ (1)

(iv) $TA \times TB - TM \times TB = TA \times MB$

$TB(TA - TM) = TA \times MB$ (1)

$TB \times AM = TA \times MB$

$\dfrac{AM}{MB} = \dfrac{TA}{TB}$

i.e. M divides AB internally in the same ratio as T divides BA externally (1)

(c)(i) $I_n = \int \tan^n dx$

$= \int \tan^{n-2}(\sec^2 x - 1)\,dx$ (1)

$= \int \tan^{n-2}\sec^2 x - \tan^{n-2}\,dx$ (1)

$= \dfrac{\tan^{n-1}}{n-1} - I_{n-2}$ (1)

(ii) $I_6 = \frac{1}{5}\tan^5 x - I_4$

$= \frac{1}{5}\tan^5 x - \left[\frac{1}{3}\tan^3 x - I_2\right]$

$= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - I_0$ (1)

$= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x + c$ (1)

Q6(a) $\delta V = \frac{1}{2} \cdot 2y \cdot \delta x$

$= \frac{1}{2}(4-x^2)\sqrt{4-x^2}\,\delta x$

$= \frac{1}{2}(4-x^2)^{3/2}\,\delta x$

$V = \lim_{\delta x \to 0} \sum_{x=-2}^{2} \frac{1}{2}(4-x^2)^{3/2}\,\delta x$

$= \int_{-2}^{2} \frac{1}{2}(4-x^2)^{3/2}\,dx$ (1)

$= \int_0^2 (4-x^2)^{3/2}\,dx$ [$(4-x^2)^{3/2}$ is an even function]

Let $x = 2\sin\theta$ $\dfrac{dx}{d\theta} = 2\cos\theta$

$= 2\sqrt{1 - \left(\frac{x}{2}\right)^2}$

$= \sqrt{4 - x^2}$ (1)

$4 - x^2$

$= 4(1 - \sin^2\theta)$

$= 4\cos^2\theta$.

$x = 0$, $2\sin\theta = 0 \Rightarrow \theta = 0$

$x = 2$, $2\sin\theta = 2 \Rightarrow \theta = \frac{\pi}{2}$. (1)

$V = \int_0^{\frac{\pi}{2}} (4\cos^2\theta)^{3/2}\, 2\cos\theta\, d\theta$.

$= 16\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta$.

$= 16\int_0^{\frac{\pi}{2}} \left(\dfrac{\cos 2\theta + 1}{2}\right)^2 d\theta$.

$= 4\int_0^{\frac{\pi}{2}} (\cos^2 2\theta + 2\cos 2\theta + 1)\,d\theta$

$= 4\int_0^{\frac{\pi}{2}} \dfrac{\cos 4\theta + 1}{2} + 2\cos 2\theta + 1\, d\theta$

$= 2\int_0^{\frac{\pi}{2}} \cos 4\theta + \cos 2\theta + 3\, d\theta$.

$= 2\left[\dfrac{\sin 4\theta}{4} + \dfrac{\sin 2\theta}{2} + 3\theta\right]_0^{\frac{\pi}{2}}$

$= 2\left[\left(0 + 0 + \frac{3\pi}{2}\right) - (0 + 0 + 0)\right]$

$= 3\pi$ unit^3.

(b)(i) $xy = 1$ $\dfrac{dy}{dx} = -\dfrac{1}{x^2}$

At (x_1, y_1) $m_T = -\dfrac{1}{x_1^2}$

Eqn of tangent at (x_1, y_1) is

$y - y_1 = -\dfrac{1}{x_1^2}(x - x_1)$

$x_1 y - x_1 y_1 = -\dfrac{x}{x_1} + 1$ (1)

But $x_1 y_1 = 1$ + $y_1 = \dfrac{1}{x_1}$

$\therefore x_1 y - 1 = -x y_1 + 1$

$x_1 y + x y_1 = 2$. (1)

(ii) Tangents at P + Q are

$x_1 y_0 + x_0 y_1 = 2$

+ $x_2 y_0 + x_0 y_2 = 2$ (1)

Eqn $xy_0 + x_0 y = 2$ satisfied by both ~~tangent~~ P + Q.

$\therefore xy_0 + x_0 y = 2$ is eqn. of PQ. (1)

(iii) x_1, x_2 are solution to the set of simultaneous eqns

$xy_0 + x_0 y = 2$ — (1)

and $xy = 1$ — (2)

Sub $y = \frac{1}{x}$ into (1)

$xy_0 + \dfrac{x_0}{x} = 2$

$x^2 y_0 + x_0 = 2x$

$x^2 y_0 - 2x + x_0 = 0$. (1)

(iv) $R \equiv \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

from (iii) $x_1 + x_2 = \dfrac{2}{y_0}$ (sum of roots)

$\therefore \dfrac{x_1 + x_2}{2} = \dfrac{1}{y_0}$

Sub into eqn of PQ

$\frac{1}{y_0} \cdot y_0 + x_0 y = 2$

$1 + x_0 y = 2$

$y = \frac{1}{x_0}$

$\therefore R \equiv \left(\frac{1}{y_0}, \frac{1}{x_0}\right)$

Q6(c)(v) If T moves on $xy = c^2$.

then $x_0 y_0 = c^2$.

For R $x = \frac{1}{y_0}$, $y = \frac{1}{x_0}$

$xy = \frac{1}{x_0 y_0} = \frac{1}{c^2}$.

$\therefore$ R moves on $xy = \frac{1}{c^2}$ which is a hyperbola.

Q7(a)(i) $\tan\left(A + \frac{\pi}{2}\right)$

$= -\tan\left(\pi - \left(A + \frac{\pi}{2}\right)\right)$

$= -\tan\left(\frac{\pi}{2} - A\right) = -\cot A$ ①

ii) $\tan\left[(2n+1)\frac{\pi}{4}\right] = (-1)^n$.

For $n=1$ LHS $= \tan\frac{3\pi}{4} = -1$

RHS $= (-1)^1 = -1 = $ LHS.

$\therefore$ True for $n=1$.

Assume true for $n=k$

$\tan\left[(2k+1)\frac{\pi}{4}\right] = (-1)^k$.

For $n = k+1$, need to prove

$\tan\left[(2k+3)\frac{\pi}{4}\right] = (-1)^{k+1}$

LS $= \tan\left[(2k+3)\frac{\pi}{4}\right]$

$= \tan\left[(2k+1)\frac{\pi}{4} + \frac{\pi}{2}\right]$

$= -\cot\left[(2k+1)\frac{\pi}{4}\right]$ ①

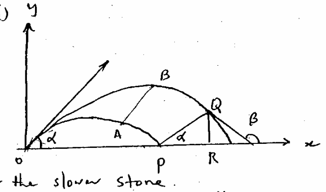
$= -\frac{1}{\tan\left[(2k+1)\frac{\pi}{4}\right]} = -\frac{1}{(-1)^k} = (-1)^{k+1} =$ RHS ①

$\therefore$ True for $n=k+1$ if true for $n=k$.

$\because$ True for $n=1$, hence true for $n = 2, 3, 4, \ldots$

$\therefore$ True for $n \geq 1$.

(b)(i)



For the slower stone.

$\ddot{y}_1 = -g$ $\qquad \ddot{x}_1 = 0$

$\dot{y}_1 = -gt + U\sin\alpha$ $\qquad \dot{x}_1 = U\cos\alpha$

$y_1 = -\frac{1}{2}gt^2 + Ut\sin\alpha$ $\qquad x_1 = Ut\cos\alpha$.

Similarly for the faster stone

$y_2 = -\frac{1}{2}gt^2 + Vt\sin\alpha$, $x_2 = Vt\cos\alpha$ ①

At time t, let the stones be at A & B.

$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}$ ①

$= \frac{-\frac{1}{2}gt^2 + Vt\sin\alpha - (-\frac{1}{2}gt^2 + Ut\sin\alpha)}{Vt\cos\alpha - Ut\cos\alpha}$

$= \frac{t\sin\alpha(V-U)}{t\cos\alpha(V-U)} = \tan\alpha$. ①

(ii) When the faster stone is at Q, the top of the wall, the slower stone is at P.

$m_{PQ} = \tan\alpha$, $\therefore \angle QPR = \alpha$. ①

$\tan\alpha = \frac{QR}{PR} = \frac{h}{PR}$

$\therefore PR = \frac{h}{\tan\alpha} = h\cot\alpha$. ①

(iii) When $y_1 = 0$.

$-\frac{1}{2}gt^2 + Ut\sin\alpha = 0$.

$gt^2 = 2Ut\sin\alpha$

$t = \frac{2U\sin\alpha}{g}$ $\quad (t \neq 0)$.

At this instant, the faster stone is at Q.

$\tan(\pi - \beta) = \frac{\dot{y}}{\dot{x}}$

$\therefore \tan\beta = -\frac{\dot{y}}{\dot{x}}$ ①

$\tan\beta = \frac{-(-gt + V\sin\alpha)}{V\cos\alpha}$

$= \frac{\cancel{g}\left(\frac{2U\sin\alpha}{\cancel{g}}\right) - V\sin\alpha}{V\cos\alpha}$.

$= \frac{\sin\alpha(2U - V)}{V\cos\alpha} = \frac{\tan\alpha(2U-V)}{V}$ ①

$V\tan\beta = \tan\alpha(2U - V)$

$V(\tan\alpha + \tan\beta) = 2U\tan\alpha$ ①

(iv) If $\beta = \frac{1}{2}\alpha$.

$\tan\alpha = \frac{2\tan\beta}{1-\tan^2\beta}$

$V\left(\frac{2\tan\beta}{1-\tan^2\beta} + \tan\beta\right) = 2U\left(\frac{2\tan\beta}{1-\tan^2\beta}\right)$ ①

$2V\tan\beta + V\tan\beta(1-\tan^2\beta) = 4U\tan\beta$.

$2V + V(1 - \tan^2\beta) = 4U$.

$U = \frac{1}{4}V(3 - \tan^2\beta)$

$< \frac{3}{4}V$ $\quad (\because \tan^2\beta > 0)$ ①

Q8(a) $P = z$, $S = 1$, $S' = -1$.

$PS = |z-1|$, $PS' = |z+1|$

If $|z-1| + |z+1| = 4$, then

$PS + PS' = 4$.

The locus of P is therefore an ellipse with foci S, S' and major axis 4 units. ①

i.e. $a = 2$.

$ae = 1$ $\qquad \therefore e = \frac{1}{2}$. ①

$b^2 = a^2(1 - e^2) = 3$

$\therefore$ Cartesian eqn of P is

$\frac{x^2}{4} + \frac{y^2}{3} = 1$ ①

(b) $\uparrow m\ddot{x}$ $\quad \downarrow mg$ $\quad \downarrow \frac{mv}{k}$ $\qquad v_0 = k(h - g)$

(i) $m\ddot{x} = -mg - \frac{mv}{k}$.

$\ddot{x} = -\left(\frac{kg + v}{k}\right)$. ①

(ii) $\frac{dv}{dt} = -\left(\frac{kg+v}{k}\right)$

$\frac{dt}{dv} = \frac{-k}{kg+v}$.

$t = -k\int_{v_0}^{v} \frac{1}{kg+v}\,dv$.

$= -k\left[\ln(kg+v)\right]_{v_0}^{v}$ ①

$= -k\left[\ln(kg+v) - \ln(kg+v_0)\right]$

$= -k\ln\left(\frac{kg+v}{kg+v_0}\right)$

$= k\ln\left(\frac{kg+v_0}{kg+v}\right)$

$= k\ln\left(\frac{\cancel{kg} + hk - \cancel{kg}}{kg+v}\right)$

$= k\ln\left(\frac{hk}{kg+v}\right)$ ①

(iii) Max. height when $v = 0$.

$t = \ln\left(\frac{kh}{kg}\right) = k\ln\left(\frac{h}{g}\right)$. ①

$v\frac{dv}{dx} = -\left(\frac{kg+v}{k}\right)$

$\frac{dx}{dv} = \frac{-kv}{kg+v}$.

$x = -k\int \frac{v}{kg+v}\,dv$.

$= -k\int\left(1 - \frac{kg}{kg+v}\right)dv$.

$= -k\left[v - kg\ln(kg+v)\right] + c$. ①

When $x=0$ $\quad v = kh - kg$

$0 = -k(kh - kg - kg\ln(\cancel{kg} + kh - \cancel{kg})] + c$

$c = (kh - kg - kg\ln(kh))k$.

Q8(b)(iii) Cont.

$x = -k\left[v - kg\ln(kg+v)\right] + k\left[kh - kg - kg\ln(kh)\right]$

$= k\left[kh - kg - v + kg\ln\left(\frac{kg+v}{kh}\right)\right]$

At $x = H$ $\quad v = 0$.

$H = k\left[kh - kg + kg\ln\left(\frac{g}{h}\right)\right]$ ①

(c)(i) $P(x) = (x^2 - a^2)Q(x) + px + q$.

$P(a) = pa + q$ — ①

$P(-a) = -pa + q$ — ②.

① − ②

$P(a) - P(-a) = 2pa$.

$\therefore p = \frac{1}{2a}\left[P(a) - P(-a)\right]$ ①

① + ②

$P(a) + P(-a) = 2q$

$\therefore q = \frac{1}{2}\left[P(a) + P(-a)\right]$ ①

(ii) $x^n - a^n = (x^2 - a^2)Q(x) + px + q$

$P(x) = x^n - a^n$

When n is even $P(a) = 0$

$P(-a) = 0$.

$\therefore q = 0$, $p = 0$.

$\therefore$ no remainder. ①

When n is odd

$P(a) = 0$ $\quad P(-a) = -2a^n$. ①

$px + q = \frac{1}{2a}\left[2a^n\right]x + \frac{1}{2}(0 - 2a^n)$

$= a^{n-1}x - a^n$. ①