STUDENT NUMBER ....

Class ....

NORMANHURST BOYS' HIGH SCHOOL
NEW SOUTH WALES

2013
Higher School Certificate
Trial Examination

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown in every question

Total Marks – 100

Section I Pages 2–5

10 marks

- Attempt Questions 1–10
- Allow about 20 minutes for this section

Section II Pages 5–11

90 marks

- Attempt Questions 11–16
- Allow about 2 hours 40 minutes for this section

Assessable Outcomes: A student

O1	applies graphical methods to various functions & solves polynomials.
O2	applies a wide variety of techniques involving integration.
O3	applies problem solving techniques with complex numbers.
O4	solves conics & determines volumes by methods of integration.
O5	solves restricted motion problems in mechanics & extension 1 harder topics.

TIE ANSWER SHEET TO THE QUESTION PAPER AND YOUR WRITING PAPER.

HAND UP IN ONE TIED BUNDLE.

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Section I

10 marks
Attempt Questions 1–10
Allow about 20 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 The equation of the tangent to $xy^3 + 2y = 4$ at the point (2, 1) is

(A) $x + 8y = 10$

(B) $x - 8y = 10$

(C) $x + 8y = -10$

(D) $x - 8y = -10$

2 If $z = 1 - \sqrt{3}\,i = 2\left(\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right)$, then what is the value of z^{21}?

(A) 2^{21}

(B) -2^{21}

(C) $(2^{21})i$

(D) $-(2^{21})i$

3 When the circle $|z - (3 + 4i)| = 5$ is sketched on the Argand Diagram the maximum value of $|z|$ occurs when z lies at the end of the diameter that passes through the centre and the origin.

What is the maximum value of $|z|$?

(A) $\sqrt{5}$

(B) 5

(C) 10

(D) $\sqrt{10}$

4 One rational root exists for $P(x) = 2x^3 - 3x^2 + 4x + 3$ such that $P(\frac{-1}{2}) = 0$.

When $P(x)$ is fully factorised over the complex field, what is the result?

(A) $(2x+1)(x^2 - 2x + 3)$

(B) $(2x+1)(x - 1 + i\sqrt{2})(x + 1 + i\sqrt{2})$

(C) $(2x+1)(x + 1 - i\sqrt{2})(x + 1 + i\sqrt{2})$

(D) $(2x+1)(x - 1 - i\sqrt{2})(x - 1 + i\sqrt{2})$

5 The cubic equation $2y^3 - 9y^2 + 12y + k = 0$ has two equal roots.

What are the possible values for k?

(A) -4 and -5

(B) -4 and 5

(C) 4 and -5

(D) 4 and 5

6

Which of the following graphs is the locus of the point P representing the complex number z moving in an Argand diagram such that $|z - 2i| = 2 + \text{Im}\, z$?

(A) a circle

(B) a parabola

(C) a hyperbola

(D) a straight line

7 What is the area bounded by the x axis and the curve $y = x(16 + x^2)^{-0.5}$ between $x = 0$ and $x = 3$?

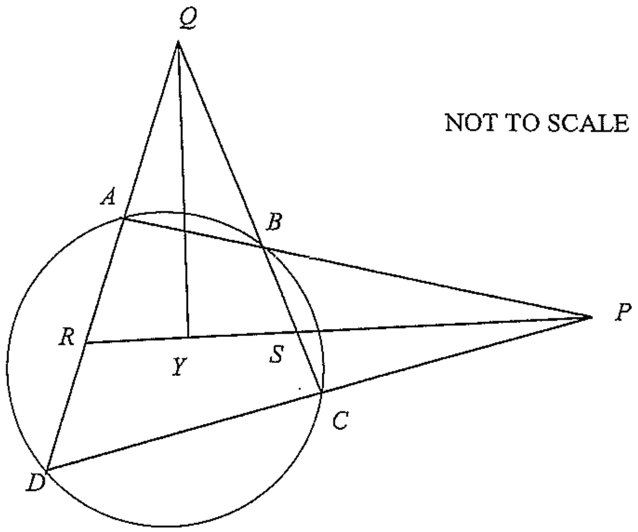
(A) $3\ units^2$
(B) $log_e 3\ units^2$
(C) $log_e e\ units^2$
(D) $log_e 1\ units^2$

8 For constant k, the equation $e^{2x} = k\sqrt{x}$ has exactly one solution when there is a common point as well as a common tangent.

What is the value of k?

(A) 1
(B) $\sqrt{e}$
(C) $2\sqrt{e}$
(D) e

9 $ABCD$ is a cyclic quadrilateral. Q and P are external points such that Y lies on the line PR and S is the intersection of PR and QC. Assume that PR bisects angle APD and QY bisects angle DQC.



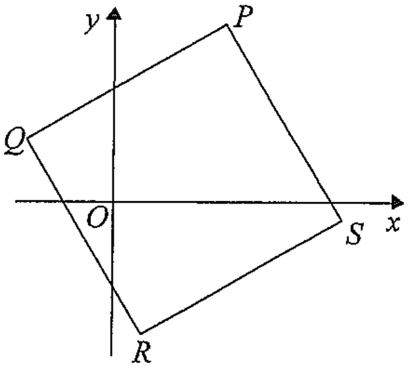
NOT TO SCALE

Which of the following is NOT true?

(A) $<QYR$ is a right angle
(B) ΔQRS is always isosceles
(C) $ABCD$ is always a kite
(D) Y is always the midpoint of RS

10

In the Argand diagram below vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$, $\overrightarrow{OR}$, $\overrightarrow{OS}$ represent the complex numbers p, q, r, s respectively where $PQRS$ is a square.



The statement $q - s = i(p - r)$ about lengths of the square is

(A) always true
(B) never true
(C) sometimes true
(D) not able to be accurately determined

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours 40 minutes for this section

All necessary working should be shown in every question.

Question 11 (15 marks) Use a SEPARATE writing page.

(a) (i) Find a primitive function for each of $\frac{x+1}{x^2+2x+5}$ and $\frac{1}{x^2+2x+5}$. **2**

(ii) Hence, or otherwise, find $\int \frac{x}{x^2+2x+5}\,dx$. **1**

(b) Evaluate $\int_1^3 \frac{dx}{x(x+2)}$. **3**

(c) (i) Express $(sec x\ tan\, x)^4$ as a product involving $sec^2 x$. **1**

(ii) Show that $\int_0^{\frac{\pi}{4}} sec^4 x\ tan^4 x\, dx = \frac{12}{35}$. **2**

(d) Use the t-substitution method with $t = tan\frac{\theta}{2}$ to find the value of **3**

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{1+sin\,\theta+\cos\theta}.$$

(e) (i) Show that $U_n = \frac{n-1}{n} U_{n-2}$ if $U_n = \int_0^{\frac{\pi}{2}} sin^n x.$ **2**

(ii) Hence, or otherwise, prove that $k = 32$ when $U_4 - U_6 = \frac{\pi}{k}$. **1**

Question 12 (15 marks) Use a SEPARATE writing page.

(a) (i) Write $\frac{3}{x+2} + x - 2$ as a single algebraic fraction. **1**

(ii) Sketch $y = \frac{3}{x+2} + x - 2$. **1**

(iii) Hence, or otherwise, solve the inequality $\frac{x^2-1}{x+2} \leq 0$. **1**

(b) The roots of $x^3 + 2x^2 - 3x - 1 = 0$ are α, β and γ. **4**

Find an equation whose roots are $\frac{\alpha\beta}{\gamma}$, $\frac{\alpha\gamma}{\beta}$ and $\frac{\beta\gamma}{\alpha}$.

(c) The points, $P(cp, cp^{-1})$ and $Q(cq, cq^{-1})$, lie on the rectangular hyperbola $xy = c^2$. The chord PQ meets the x axis at C. O is the centre of the hyperbola and R is the midpoint of PQ.

(i) Draw a sketch showing all the information. **1**

(ii) Find the equation of chord PQ. **2**

(iii) Find the co-ordinates of C. **1**

(iv) Find the co-ordinates of R. **1**

(v) Show that OR = RC. **3**

Question 13 (15 marks) Use a SEPARATE writing page.

(a) The hyperbola, H has the Cartesian equation $5x^2 - 4y^2 = 20$.

P is an arbitrary point, $(2\sec\theta, \sqrt{5}\tan\theta)$.

(i) Find the eccentricity of H and state the co-ordinates of its foci, S and S'. **2**

(ii) State the equations of the directrices and both asymptotes for H. **2**

(iii) Sketch the curve, clearly showing all of the above features. **1**

(iv) Demonstrate that $P(2\sec\theta, \sqrt{5}\tan\theta)$ lies on H. **1**

(v) Show that the tangent to H at P is **2**

$$\frac{x\sec\theta}{2} - \frac{y\tan\theta}{\sqrt{5}} = 1.$$

(vi) The tangent at P cuts the asymptotes at L and M. **3**

Prove that $LP = PM$.

(vii) O is the origin. **2**

Show that the area of ΔOLM is independent of the position P on H.

(b) The function $y = f(x)$ is denoted by $f(x) = x^3 - 6x$.

(i) Sketch the graph of $y = |f(x)| = |x^3 - 6x|$ on a separate set of axes. **1**

(ii) Sketch the graph of $y = \frac{1}{f(x)} = (x^3 - 6x)^{-1}$ on a separate set of axes. **1**

Question 14 (15 marks) Use a SEPARATE writing page.

(a) Consider the region bounded by the two curves $y = 3 - x^2$ and $y = -2x$.

Suppose two vertical lines, one unit apart, intersect the given region.

(i) The vertical lines are $x = x_1$ and $x = x_1 + 1$. **4**

Find the value/s of x_1 so that the area enclosed by the two vertical lines and the two curves is a maximum.

(ii) Show that this enclosed area is $3\frac{11}{12}$ units^2. **2**

Justify that this area is the maximum.

(b) The area bounded by the y axis, the line $y = 1$ and $y = \sin x$ is revolved about the line $y = 1$. **4**

Using a slicing technique, find the volume of the solid of revolution formed between $x = 0$ and $x = \frac{\pi}{2}$.

(c) Use the method of cylindrical shells to find the volume of the solid formed when the area enclosed by $y = (x-2)^2$ and $y = 4$ is rotated about the y axis. **5**

Question 15 (15 marks) Use a SEPARATE writing page.

(a) (i) Factorise $z^5 + 1$ over the real field. 1

(ii) List the roots of $z^5 + 1 = 0$ in $rcis\theta$ form. 1

(iii) Deduce that $2\cos\frac{\pi}{5} + 2\cos\frac{3\pi}{5} - 1 = 0$. 2

(b) (i) Using the tan (A – B) expansion, show that if $mx = \tan^{-1}Q - \tan^{-1}v$

then $mx = \tan^{-1}(\frac{Q-v}{1+Qv})$. 1

(ii) Show that $a = 1$, $b = -1$ and $c = 0$ if $\frac{1}{v+v^3} = \frac{a}{v} + \frac{bv+c}{1+v^2}$. 1

(c)

A particle moves in a straight line against a resistance numerically equal to $m(v + v^3)$ where v is its velocity. Initially the particle is at the origin and is travelling with velocity Q, where $Q > 0$. Assume $\ddot{x} = -m(v + v^3)$.

(i) Show that the displacement x in terms of v is $x = \frac{1}{m}\tan^{-1}(\frac{Q-v}{1+Qv})$. 3

(ii) Prove that $t = \frac{1}{2m}\log_e(\frac{Q^2(1+v^2}{v^2(1+Q^2)})$ where t is the time elapsed. 3

(iii) Find an expression for the square of the velocity as a function of time. 1

(iv) By finding the limiting values of velocity and displacement, explain why this particle eventually slows down and show that this occurs near a point where $Q = \tan(mx)$. 2

Question 16 (15 marks) Use a SEPARATE writing page

(a) A sequence of polynomials, called the *Bernoulli Polynomials*, is defined by the three conditions:-

1. $B_0(x) = 1$
2. $B_n'(x) = \frac{d}{dx}(B_n(x)) = nB_{n-1}(x)$
3. $\int_0^1 B_n(x)dx = 0$ if $n \geq 1$

(i) Show that $B_1(x) = x - \frac{1}{2}$. **3**

(ii) If $B_n(x+1) - B_n(x) = nx^{n-1}$ and $g(x) = B_{n+1}(x+1) - B_{n+1}(x)$, prove that **2**

$$g'(x) = (n+1)nx^{n-1}.$$

Hence show $g(x) = (n+1)x^n + C$, where C is a constant.

(iii) Use the method of mathematical induction to prove that **5**

$$B_n(x+1) - B_n(x) = nx^{n-1} \quad \text{if } n \geq 1.$$

(b) (i) By squaring, or otherwise, show that for $k \geq 0$, **1**

$$2k + 3 > 2\sqrt{k+2}\sqrt{k+1}.$$

(ii) By decomposing $2k + 3$ and factorising $2\sqrt{k+2}\sqrt{k+1} - 2(k+1)$ show that for $k \geq 1$, **2**

$$\frac{1}{\sqrt{k+1}} > 2(\sqrt{k+2} - \sqrt{k+1}).$$

(iii) Hence, or otherwise, show for $n \geq 1$, **2**

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots\dots\dots + \frac{1}{\sqrt{n}} > 2(\sqrt{n+1} - 1).$$

End of paper