MOOREFIELD GIRLS' HIGH SCHOOL

2005
TRIAL HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics

Extension 2

General Instructions

- Reading Time- 5 minutes
- Working Time – 3 hours
- Write using a blue or black pen
- Approved calculators may be used
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown for every question.

Total marks (120)

- Attempt Questions 1-8
- All questions are of equal value
- Start a fresh sheet of paper for each question.
- Put your name and the question number at the top of each sheet.

Question 1 (15 marks) **Marks**

(a) Find $\int x\cos(x^2)e^{\sin(x^2)}\,dx$ **2**

(b) By completing the square in the denominator, find

$$\int \frac{3\,dx}{x^2-6x+13}$$ **3**

(c) Apply the method of partial fractions to show that

$$\int_3^5 \frac{3x+2}{x^2-4}\,dx = \ell n\left(\frac{63}{5}\right)$$ **4**

(d) Using integration by parts, evaluate

$$\int_0^1 \sin^{-1}x\ dx$$ **3**

(e) Find $\int \frac{2x^2+4x-3}{x+1}\,dx$ **3**

Question 2 (15 marks)

(a) Given the complex numbers $A = 3-4i$ and $B = 1+i$, determine the following in the form $x+iy$

(i) ✓ $A-B$ **1** ✓

(ii) ✓ $\dfrac{A}{B}$ **2** - 1

(iii) ✓ $\sqrt{A}$ **3** ✓

(b) Given $C = 1+\sqrt{3}i$

(i) ✓ Write C in mod-arg form **2** ✓

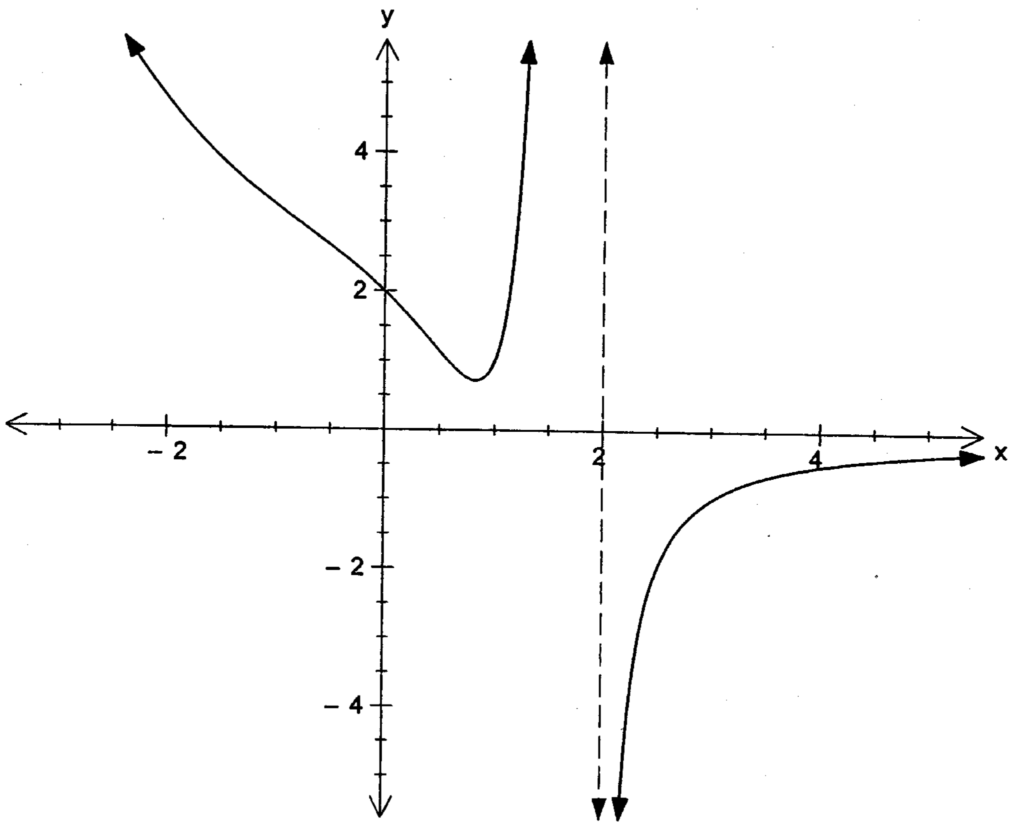
(ii) ✓ Hence, using De Moivre's theorem, find C^6 **2** ✓

(c) ✓ Determine the Cartesian equation of the locus of $Z = x+iy$ given that $\arg(Z-1) - \arg(Z+1) = \dfrac{\pi}{2}$ **3** - 3

(d) ✓ On an Argand diagram, sketch the region where the inequalities $1 \le \operatorname{mod} Z \le 3$ and $0 \le \arg Z \le \dfrac{\pi}{4}$ both hold. **2** ✓

Question 3 (15 marks)

(a) The diagram below shows the graph of $y = f(x)$



Draw separate sketches of the following (indicate important features)

(i) $y = \dfrac{1}{f(x)}$ **2** – 1

(ii) $y = [f(x)]^2$ **2** ✓

(iii) ✓ $y = \int f(x)dx$ given that when $x = 0, y = 0$ **2** – 1

Question 3 is continued on the next page

TP

Question 3 continued.

Marks

(b) Given that α, β and γ are the roots of the equation $3x^3 + 2x^2 - 5x + 1 = 0$, find equations whose roots are:

(i) α^2, β^2 and γ^2 **3** −2

(ii) $\dfrac{1}{\alpha}, \dfrac{1}{\beta}$ and $\dfrac{1}{\gamma}$ **2**

(c) Using your answer to (b) (i) above, or otherwise, evaluate $\alpha^3 + \beta^3 + \gamma^3$. **4** −4

$\alpha^2\beta^2 + \alpha^2\gamma^2 + \gamma^2\beta^2 + \alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\gamma^2\beta -$

$\alpha^2\beta\gamma$

Question 4 (15 marks)

(a) A solid has its base in the XY plane being the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$
Cross-sections perpendicular to the major axis are squares.

(i)√ Show that the area of the cross-section at $x = p$ is given by $36-\frac{9p^2}{4}$ units ^{2} **3** /

√

(ii) Hence by using the method of slicing, calculate the volume of the solid. **2** – 1

(b) $P\left(cp,\frac{c}{p}\right)$ and $Q\left(cq,\frac{c}{q}\right)$ are two variable points on the rectangular hyperbola $xy=c^2$ which move so that the points P, Q and $S\left(c\sqrt{2},c\sqrt{2}\right)$ are always collinear. The tangents to the hyperbola at P and Q intersect at the point R.

(i)√ Show that the tangent to the hyperbola $xy=c^2$ at the point $T\left(ct,\frac{c}{t}\right)$ has √equation $x+t^2y=2ct$. **2** /

(ii) Hence show that R has coordinates $\left(\frac{2cpq}{p+q},\frac{2c}{p+q}\right)$. **2** /

(iii) Show that $p+q=\sqrt{2}\,(1+pq)$. / **3** – 3

(iv)√ Hence find the equation of the locus of R. **3** – 2

$2 \div \frac{\sqrt{2}}{2}$

$\frac{2}{\sqrt{2}}$

Question 5 (15 marks) **Marks**

(a) A solid of mass m kg is falling under gravity in a fluid in which the resistance is proportional to the velocity ($V\ ms^{-1}$).

(i) Write down the equation for this motion. **1** √

(ii) If the solid is initially at rest show that, after t seconds the velocity is given by $V = \frac{g}{k}(1 - e^{-kt})$ **4** -2

(iii) What would be the terminal velocity of this solid? **1** -½

(iv) Find an expression for the distance fallen, in terms of V **4** -1

(b) A solid of mass 2kg is attached to an inextensible string of length 1•5 metres, the other end of the string being fixed. The mass rotates in a horizontal circle with an angular velocity of $\pi\ rad\ s^{-1}$, forming a conical pendulum. (Take $g = 10ms^{-2}$)

(i) Calculate the tension in the string **3** -2

(ii) Determine the angle between the string and the vertical axis **1** -1

(iii) Find the radius of the rotation **1** -1

$T = \tan\theta$

$\frac{T\sin\theta}{T\cos\theta} = 1.$

Question 6 (15 marks)

(a) The sequence of numbers $U_1, U_2, U_3, \ldots$ is defined by $U_{n+2} = 5U_{n+1} - 6U_n$ **5** -2
If $U_1 = 1$ and $U_2 = 5$, prove by Induction that $U_n = 3^n - 2^n$

(b) If Z_1, Z_2, Z_3, Z_4 and Z_5 are the roots of $Z^5 = 1$

(i) Determine the values of all of these roots, in the form $\cos\theta + i\sin\theta$ **2**

(ii) Factorise $Z^5 = 1$ in terms of quadratic and linear factors. **2** -2

(iii) Hence show that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = \frac{-1}{2}$ **2** -2

(c) If $Z = \cos\theta + i\sin\theta$

(i) Show that $Z^n + \frac{1}{Z^n} = 2\cos n\theta$ **1**

(ii) Hence by expanding $(Z + \frac{1}{Z})^4$, **3**
find an expression for $\cos^4\theta$ in the form $a\cos 4\theta + b\cos 2\theta + c$

cos 72.

cos 144

cos 144.

cos 72.

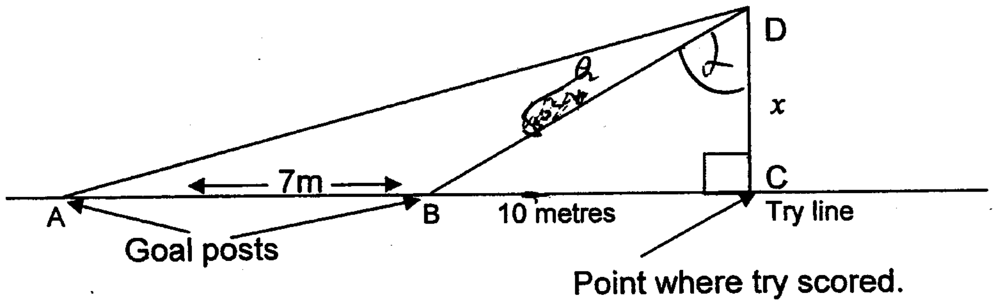
Question 7 (15 marks)

(a) A corner on a racing circuit is the arc of a circle of radius 50 metres. Calculate the angle at which is corner must be inclined so that a vehicle of mass 500kg can take this corner at a speed of $30ms^{-1}$ without any tendency to slip sideways up or down the track. (Take $g = 9\cdot 8ms^{-2}$) **4**

(b) Given that the roots of the equation $x^3 + ax^2 + bx + c = 0$ form a geometric sequence, determine the relationship between a, b and c **4**

(c) In a game of rugby, the goal posts (A and B below) are approximately 7 metres apart.
When a try is scored, a kick at goal is allowed from any point on the line which is perpendicular to the goal line at the point where the try is scored (the point C below)..

A try is scored 10 metres to the side of the goal post B. The kicker takes the ball back a distance of x metres, giving an angle of θ within which he must kick the ball to go between the posts.

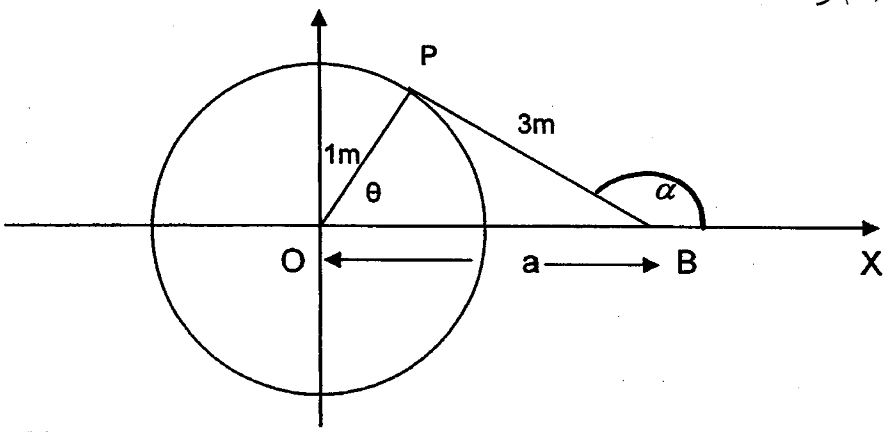


(i) Show that $\tan\theta = \dfrac{7x}{170 + x^2}$ **4**

(ii) Hence show that the kicker must take the ball approximate 13 metres to maximise the angle between the posts for a successful kick. **3**

Question 8 (15 marks) **Marks**

(a)



A rod OP is such that P is free to move in a circle as shown. Rod PB is joined to OP with a flexible joint at P with B restricted to move along the X axis. P is moving about O with angular velocity of π rad s^{-1}. The length of OP is 1 metre and PB is 3 metres.

(i) By letting $\angle XBP = \alpha$ find an expression for the angular velocity of P about B **3**

(ii) Hence find this angular velocity when $\theta = \frac{\pi}{4}$ **3**

(iii) If the length of OB is 'a' metres, find an expression for the length a and hence the velocity of Point B along the X axis $\left(\frac{da}{dt}\right)$ when $\theta = \frac{\pi}{4}$ **5**

(b) If $\alpha\beta\gamma$ represents a 3 digit number and $\alpha + \beta + \gamma$ is divisible by three show that this number is divisible by 3. **2**

(c) By taking the log of both sides of $y = U(x).V(x)$ verify the differentiation of a product rule **2**