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## AN ELEMENTARY PROOF OF THE FORMULA $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$

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The formula in the title has been well known, but its various known proofs are less elementary, (see for example, [1], p. 219, 360, [3], p. 237, 267, 324, [4], Problem 99, p. 196, [5], p. 379). The following proof is quite elementary in character.

For any positive integer $n$, we consider

$$\int_0^{\pi/2} \cos^{2n} t\,dt.$$

Applying integration by parts twice, we obtain

$$\int_0^{\pi/2} \cos^{2n} t\,dt = [t \cos^{2n} t]_0^{\pi/2} + 2n \int_0^{\pi/2} t \cos^{2n-1} t \sin t\,dt$$

$$= n[t^2 \cos^{2n-1} t \sin t]_0^{\pi/2} - n \int_0^{\pi/2} t^2[-(2n-1)\cos^{2n-2} t \sin^2 t + \cos^{2n} t]\,dt$$

$$= -2n^2 \int_0^{\pi/2} t^2 \cos^{2n} t\,dt + n(2n-1) \int_0^{\pi/2} t^2 \cos^{2n-2} t\,dt$$

$$= -2n^2 I_{2n} + n(2n-1) I_{2n-2},$$

where $I_{2n} = \int_0^{\pi/2} t^2 \cos^{2n} t\,dt$. Hence

$$-2n^2 I_{2n} + n(2n-1) I_{2n-2} = \int_0^{\pi/2} \cos^{2n} t\,dt.$$

As is known, (see, for example, [2], p. 226)

$$\int_0^{\pi/2} \cos^{2n} t\,dt = \frac{(2n-1)!!}{(2n)!!} \frac{\pi}{2},$$

where, as usual,

$$(2n)!! = 2\cdot 4 \cdots (2n-2)(2n), \quad 0!! = 1;$$
$$(2n+1)!! = 1\cdot 3 \cdots (2n-1)(2n+1), \quad (-1)!! = 1.$$

Thus we have

$$-2n^2 I_{2n} + n(2n-1)I_{2n-2} = \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2},$$
$$\frac{(2n)!!}{(2n-1)!!}I_{2n} - \frac{(2n-2)!!}{(2n-3)!!}I_{2n-2} = -\frac{\pi}{4}\frac{1}{n^2}.$$

This implies that

$$\frac{(2n)!!}{(2n-1)!!}I_{2n} - \frac{0!!}{(-1)!!}I_0 = \sum_{k=1}^{n}\left[\frac{(2k)!!}{(2k-1)!!}I_{2k} - \frac{(2k-2)!!}{(2k-3)!!}I_{2k-2}\right]$$
$$= -\frac{\pi}{4}\sum_{k=1}^{n}\frac{1}{k^2},$$

and hence that

$$\frac{(2n)!!}{(2n-1)!!}I_{2n} = \frac{\pi^3}{24} - \frac{\pi}{4}\sum_{k=1}^{n}\frac{1}{k^2} = \frac{\pi}{4}\left[\frac{\pi^2}{6} - \sum_{k=1}^{n}\frac{1}{k^2}\right].$$

It is sufficient to prove that

(1) $$\lim_{n\to\infty}\frac{(2n)!!}{(2n-1)!!}I_{2n} = 0.$$

Now we have

$$I_{2n} = \int_0^{\pi/2} t^2 \cos^{2n} t\,dt \leqq \left(\frac{\pi}{2}\right)^2 \int_0^{\pi/2} \sin^2 t \cos^{2n} t\,dt$$
$$= \frac{\pi^2}{4}\left[\int_0^{\pi/2}\cos^{2n} t\,dt - \int_0^{\pi/2}\cos^{2n+2} t\,dt\right]$$
$$= \frac{\pi^3}{8}\left[\frac{(2n-1)!!}{(2n)!!} - \frac{(2n+1)!!}{(2n+2)!!}\right] = \frac{\pi^3}{8}\frac{(2n-1)!!}{(2n+2)!!}.$$

Therefore

$$0 < \frac{(2n)!!}{(2n-1)!!}I_{2n} \leqq \frac{\pi^3}{8}\frac{1}{2n+2}.$$

Thus we establish (1); hence the formula is proved.

Finally, the author wishes to express his thanks to a referee for valuable criticism.

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