MY ROCK AND MY FORTRESS

MATHEMATICS EXTENSION 2

2006

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

General Instructions

Total marks - 120

- Reading time – 5 minutes
- Writing time – 3 hours
- Write using blue or black pen
- Attempt questions 1-8
- All questions are worth 15 marks
- Board-approved calculator may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown for each question

Question 1 (15 marks) **Marks**

(a) (i) Find $\int \frac{\sin^{-1} 2x}{\sqrt{1-4x^2}}\,dx$ 1

(ii) Using the substitution $u = \sqrt{1+x}$, find $\int \frac{xdx}{\sqrt{1+x}}$ 2

(ii) Find $\int \frac{\cos^5 x}{\sin^2 x}\,dx$ 2

(b) Evaluate $\int_0^2 x^3 e^{x^2}\,dx$ by writing $x^3 e^{x^2} = x^2 . x e^{x^2}$ 2

(c) By expressing $\frac{48}{x^3+64}$ as partial fractions, find $\int \frac{48}{x^3+64}\,dx$ 5

(d) (i) Show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ 1

(ii) Hence show that $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \frac{\pi}{4}$ 2

Question 2 (15 marks) **Marks**

(a) Simplfy $\dfrac{\left(\cos\dfrac{5\pi}{12}+i\sin\dfrac{5\pi}{12}\right)\left(\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}\right)}{\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}}$, giving your answer in the form $a+ib$ **2**

(b) Find all the complex numbers $z=a+ib$, where a and b are real, such that $|z|^2-7=2i(z+2)$ **4**

(c) (i) Find the three roots of $z^3-1=0$ in modulus argument form **3**

(ii) Find the area of the triangle formed by these three roots **1**

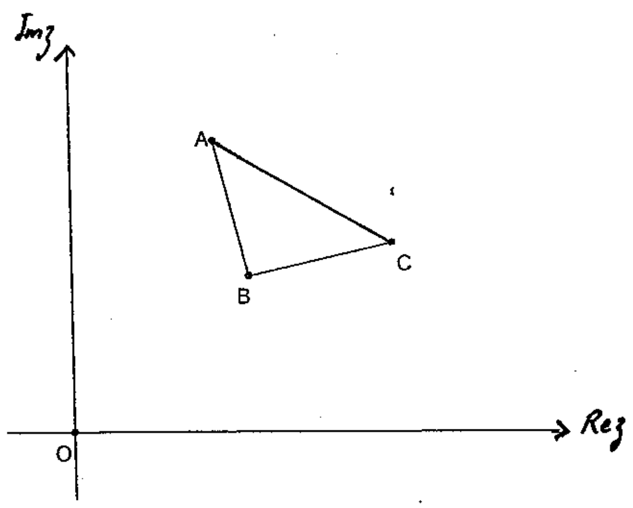
(d) By considering the complex number in the Argand plane, sketch the locus of the following on separate Argand diagrams.

(i) $\arg z=\dfrac{\pi}{3}$ **1**

(ii) $\arg \bar{z}=\dfrac{\pi}{3}$ **1**

(iii) $\arg(-z)=\dfrac{\pi}{3}$ **1**

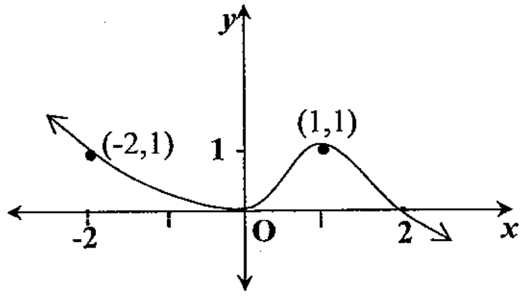
(e) **2**



In the diagram, the vertices of a triangle ABC are represented by the complex numbers z_1, z_2 *and* z_3 respectively. The triangle is isosceles and right angled at B. Show that $(z_1-z_2)^2=-(z_3-z_2)^2$

Question 3 (15 marks) **Marks**

(a) The diagram shows the graph of $y = f(x)$.
On separate diagrams sketch the graphs of the following.
Indicate clearly any asymptotes, intercepts with the axes and local maxima and minima.



(i) $y = \ln[f(x)]$ **2**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = -|f(x)|$ **2**

(iv) $y = f'(x)$ **2**

(b) (i) Given a, b and c are three non negative numbers, show that their arithmetic mean ≥ geometric mean. **3**

(ii) Hence show that $(a+b+c)\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right) \geq 9$ **1**

(c) Using induction show that for each positive integer n there are unique positive integers p_n and q_n such that $\left(1+\sqrt{2}\right)^n = p_n + q_n\sqrt{2}$. **3**

Question 4 (15 marks) **Marks**

(a) (i) By differentiation, show that $P(x)=x^5-3x^4+4x^3-4x^2+3x-1$ has $x=1$ as a root of multiplicity 3. **2**

(ii) Verify that $x=i$ is also a root of $P(x)$. **1**

(iii) Hence write $P(x)$ in factorised form over the complex numbers. **2**

(b) If p, q and r are the roots of the equation $x^3+4x^2-3x+1=0$, find the equation whose roots are $\frac{1}{p}$, $\frac{1}{q}$ and $\frac{1}{r}$ **3**

(c) Consider the polynomial $k(t)=t^4+at^3+bt^2+at+1$, where a and b are real numbers.

(i) Show that if q is a zero of $k(t)$, then $\frac{1}{q}$ is also a zero of $k(t)$ **2**

(ii) Hence, or otherwise, write down all four zeros of $k(t)$, given that $(1+i)$ is a zero of $k(t)$ **2**

(d) A polynomial $P(x)$ gives remainders -2 and 1 when divided by $2x-1$ and $x-2$ respectively.
What is the remainder when $P(x)$ is divided by $2x^2-5x+2$? **3**

Question 5 (15 marks) **Marks**

(a) The ellipse E has the equation $4x^2 + 9y^2 = 36$

(i) Write down:

1. its eccentricity **1**
2. the coordinates of its foci S and S' **1**
3. the equation of the directrix **1**
4. the length of the major axis **1**

(ii) Sketch the ellipse E. Show the x and y intercepts as well as the features found in parts 2. and 3. of part (i) above **1**

(b) The region S is enclosed by the line $x + y = a;\ a > 0$, the curve $y = x^3 - ax^2$ and the y-axis.

(i) Sketch the region S; clearly labelling its intercepts with the axes. **2**

(ii) The region S is rotated around the line $x = a$ to form a solid. Use the method of cylindrical shells to find the volume of this solid. **4**

(c) If A, B and C are the angles of a triangle, **4**

show that $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

Question 6 (15 marks) **Marks**

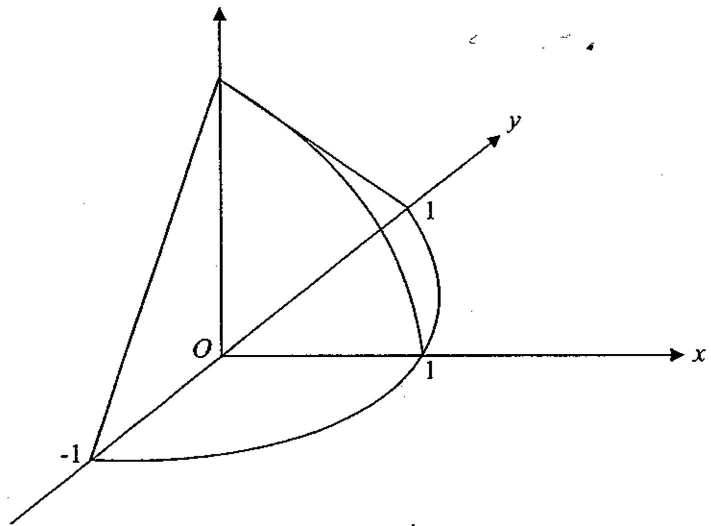
(a) (i) Show that the equation of the normal at the point P $(a\cos\theta, b\sin\theta)$ on the ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a > 0 and b > 0 is given by the equation, $xa\sin\theta - yb\cos\theta = (a^2-b^2)\sin\theta\cos\theta$. **2**

(ii) Show that the ellipse intersects the hyperbola $xy=c^2$ in four points, if $ab>2c^2$. **3**

(iii) Show that for $0<\theta<\frac{\pi}{2}$, the normal at P to the ellipse intersects the hyperbola in two points, say A and B. **3**

(iv) If M is the mid-point of AB, find the locus of M as θ varies. **3**

(b)



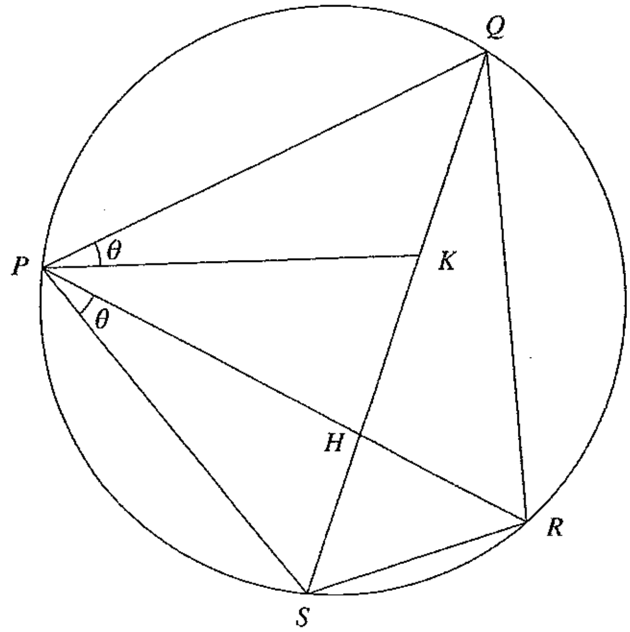
The base of a solid is the semi-circular region in the x - y plane with the straight edge running from the point $(0,-1)$ to the point $(0,1)$ and the point $(1,0)$ on the curved edge of the semicircle.
Each cross-section perpendicular to the x-axis is an isosceles triangle with each of the two equal sidelengths three quarters the length of the third side.

(i) Show that the area of the triangular cross-section at $x=a$ is $\frac{\sqrt{5}}{2}(1-a^2)$ **2**

(ii) Hence find the volume of the solid. **2**

Question 7 (15 marks) **Marks**

(a) The diagram shows a cyclic quadrilateral $PQRS$ circumscribed by a circle.
The diagonals of this quadrilateral meet at H.
K is a point on QS such that angle SPR equals angle QPK.



(i) Show that the triangles PQK and PRS are similar **2**

(ii) Show that the triangles PQR and PKS are similar **2**

(iii) Hence prove Ptolemy's theorem: **3**
"In any cyclic quadrilateral, the sum of the products of the lengths of opposite sides is equal to the product of the lengths of the diagonals."

That is prove that in this cyclic quadrilateral $PQRS$,

$$PQ \times RS + PS \times QR = PR \times QS$$

(b) A body is projected vertically upwards from the ground with initial velocity v_0 in a medium that produces a resistance force per unit mass of kv^2, where v is the velocity and k is a positive constant.

Take the acceleration due to gravity as g m s^{-2}.

(i) Prove that the maximum height H of the body above the ground is **4**
$$H = \frac{1}{2k}\log_e\left(1 + \frac{kv_0^2}{g}\right)$$

(ii) Show that in order to double the maximum height reached, **4**
the initial velocity must be increased by a factor of $(e^{2kH} + 1)^{\frac{1}{2}}$

Question 8 (15 marks) **Marks**

(a) Let $I_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^n x\, dx$ where n is a positive integer.

(i) Show that $I_n = \dfrac{1}{n-1} - I_{n-2}$ for $n \geq 2$ **3**

(ii) Hence evaluate I_5 **2**

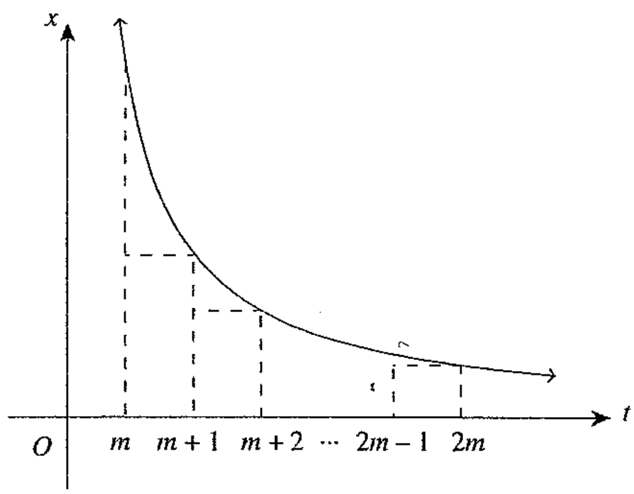
(b) (i) Prove that $\dfrac{1}{2p+1} + \dfrac{1}{2p+2} > \dfrac{1}{p+1}$, for all $p > 0$ **2**

(ii) Consider the statement **4**

$$\chi(m): \frac{1}{m+1} + \frac{1}{m+2} + \dots\dots\dots + \frac{1}{2m} \geq \frac{37}{60}$$

Show by mathematical induction that $\chi(m)$ is true for all integers $m \geq 3$

(iii) The diagram below shows the graph of $x = \dfrac{1}{t}$, for $t > 0$



1. By comparing areas, show that **1**

$$\int_m^{m+1} \frac{1}{t}\, dt > \frac{1}{m+1}$$

2. Hence, without using a calculator, show that **3**

$$\log_e 2 > \frac{37}{60}$$

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

Note: $\ln x = \log_e x, \quad x > 0$