2010

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Ext II Mathematics

General Instructions

- Reading time – 5 minutes
- Working time –3 hours
- Write using black or blue pen
- Approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown in every question
- Answer each question in a SEPARATE Writing Booklet

Total marks – 120

- Attempt Questions 1 – 8
- All questions are of equal value

Care has been taken to ensure that this paper is free of errors and that it mirrors the format and style of past HSC papers. The questions have been adapted from various sources in an attempt to provide students with exposure to a broad range of possible questions. However, there is no guarantee whatsoever that the 2007 HSC examination will have similar content, style or format. This paper is intended only as a trial for the HSC examination or as revision leading up to the examination.

Question 1 (15 marks) Use a separate page/booklet **Marks**

(a) Find $\int \frac{x}{1+x^2}dx$. **1**

(b) Find $\int \frac{10}{(x-1)(x^2+9)}dx$ **4**

(c) Find $\int \sin^3 x\, dx$. **3**

(d) Evaluate $\int_0^{\pi/2} \frac{1}{3\cos x + 4\sin x + 5}dx$. **4**

(e) Find $\int x^2 \ln x\, dx$ **3**

Question 2 (15 marks) Use a separate page/booklet

(a) Find real x and y such that $(x+iy)^2 = 3+4i$. **3**

(b) Find $\frac{z_1}{z_2}$ when $z_1 = (2+i),\ z_2 = i$ **2**

(c) Write down the moduli and arguments of $-\sqrt{3}+i$ and $4+4i$. Hence express in modulus/argument form $(-\sqrt{3}+i)(4+4i)$. **3**

(d) Complex numbers $z_1 = \frac{a}{1+i}$ and $z_2 = \frac{b}{1+2i}$ where a and b are real, are such that $z_1 + z_2 = 1$. Find a and b. **3**

(e) If $z = \cos\theta + i\sin\theta$, show that $z^n - z^{-n} = 2i\sin n\theta$. Hence show that $\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$. **4**

Question 3 (15 marks) Use a separate page/booklet **Marks**

(a) Use the graph of $f(x) = 4 - x^2$ to sketch (showing critical points) the graph of $y = |f(x)|$. **2**

(b) Sketch the graph of $y = x^2 \ln x$. **3**

(c) Show that $\dfrac{x^2}{x+1} = x - 1 + \dfrac{1}{x+1}$. Hence sketch the graph of $y = \dfrac{x^2}{x+1}$. **3**

(d) For the function $f(x) = x^2 - 1$ use the graph of $y = f(x)$ to sketch the graphs of

(i) $y = \sqrt{f(x)}$ **2**

(ii) $y^2 = f(x)$. **2**

(e) If $P(x) = 4x^3 + 15x^2 + 12x - 4$ has a double zero, find all the zeros and factorise $P(x)$ fully over the real numbers. **3**

Question 4 (15 marks) Use a separate page/booklet

(a) If $I_n = \int_1^e (\ln x)^n \, dx$ for $n \geq 0$, show that $I_n = e - nI_{n-1}$ for $n \geq 1$ and hence find I_4 **4**

(b) The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. The points T and T' are the feet of the perpendiculars from the foci S and S' respectively to this tangent.

(i) Show that $ST = \dfrac{|e\cos\theta - 1|}{\sqrt{\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2}}}$ **2**

(ii) Hence prove $ST \cdot S'T' = b^2$ **3**

(c) The equation $x^4 + 4x^3 - 3x^2 - 4x + 2 = 0$ has roots α, β, γ and δ.

Find the equations with roots

(i) $2\alpha, 2\beta, 2\gamma$ and 2δ **2**

(ii) $\dfrac{1}{\alpha}, \dfrac{1}{\beta}, \dfrac{1}{\gamma}$ and $\dfrac{1}{\delta}$ **2**

(iii) $\alpha^2, \beta^2, \gamma^2$ and δ^2 **2**

Question 5 (15 marks) Use a separate page/booklet **Marks**

(a) By taking slices perpendicular to the axis of rotation, use the method of slicing to find the volume of the solid obtained by rotating the region $\{(x, y): 0 \leq x \leq 2, 0 \leq y \leq x^3\}$ about the line $x = 2$. **4**

(b) The letters of the word "Patyaan" are arranged in a line. In how many ways can they be arranged if

(i) all the A's are together? **1**

(ii) all the A's are separated? **1**

(iii) exactly two of the A's are together? **1**

(iv) the consonants are in alphabetical order from left to right, but not necessarily together? **2**

(c) A particle of mass m is projected vertically upwards from the ground. The particle experiences a resistance of magnitude mkv^2 where k is a positive constant and the velocity of the particle has magnitude v, During its downward motion the terminal velocity of the particle is *V*. Its initial velocity of projection is half this terminal velocity.

(i) By considering the forces on the particle during its downward motion, show that $kV^2 = g$ **1**

(ii) Show that during its upward motion the acceleration of the particle is given by $V^2\ddot{x} = -g(V^2 + v^2)$, and the distance x travelled by the particle when its velocity is v is given by $x = \frac{V^2}{2g} ln\{\frac{5V^2}{4(V^2 + v^2)}\}$ **4**

(iii) Find the maximum height h of the particle above its projection point. **1**

Question 6 (15 marks) Use a separate page/booklet **Marks**

(a) The Hyperbola H has equation $xy = 16$

(i) Sketch this hyperbola and indicate on your diagram the positions and coordinates of all points at which the curve intersects the axes of symmetry. **2**

(ii) $P(4p, \frac{4}{p})$ $p > 0$ and $Q\ (4q, \frac{4}{q})$ $q > 0$ are two points on H.
Find the equation of the chord PQ. **2**

(iii) Prove that the equation of the tangent at P is $x + p^2 y = 8p$ **1**

(iv) The tangents at P,Q intersect at T. Find the coordinates of T. **2**

(v) The chord PQ produced passes through N (0,8). Find the equation of the locus of T. **2**

(b) (i) The base of a solid is the region bounded by the curves $y = e^x, y = 1 - x$ and $x = 1$.
Find the volume of the solid if every section perpendicular to x - axis is a square with one vertex on $y = e^x$ and another vertex on $y = 1 - x$. **3**

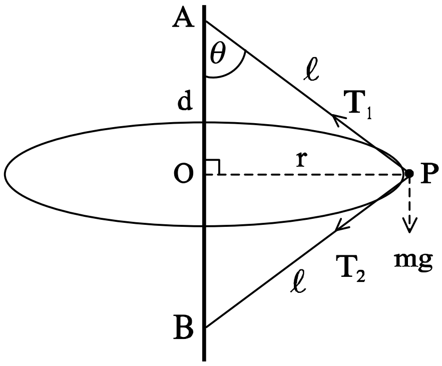
(ii) The region in (i) is rotated about the line $x = 4$. Find the volume by using cylindrical shells. **3**

Question 7 (15 marks) Use a separate page/booklet **Marks**

(a) A particle P of mass m is attached by two equal light inextensible strings each of length ℓ to two fixed points A and B on a smooth vertical rod.

Particle P is set to move in a uniform circular motion with centre O, radius r and angular velocity w.

Let d be the distance between centre O and point A, and let θ be the angle between the string AP and the rod AB.



i) Show that the tensions in the strings AP and BP are respectively **4**

$$T_1 = \frac{ml}{2}\left(w^2 + \frac{g}{d}\right) \quad \text{and} \quad T_2 = \frac{ml}{2}\left(w^2 - \frac{g}{d}\right)$$

ii) Find the angular velocity w in terms of d and g if $T_1 : T_2 = 3 : 2$. **3**

(b) Find all x such that $\sin x = \cos 5x$ and $0 < x < \pi$ **3**

(c) A machine has 6 identical wheels. In a game, a button is pushed and the 6 wheels spin. Each wheel can stop on a star or moon symbol only. The probability that a star symbol will appear on any wheel is p.

It is known that the most likely outcome for a single game on this machine is 2 stars and 4 moons and this occurs for $a < p < b$ where a and b are the lowest and highest real value boundaries for p.

Calculate the values of a and b. **5**

Question 8 (15 marks) Use a separate page/booklet **Marks**

(a) The curve $y = f(x)$ concave up for all real values of x.
a, b, c, and d are the x coordinate of four points lying on this curve.

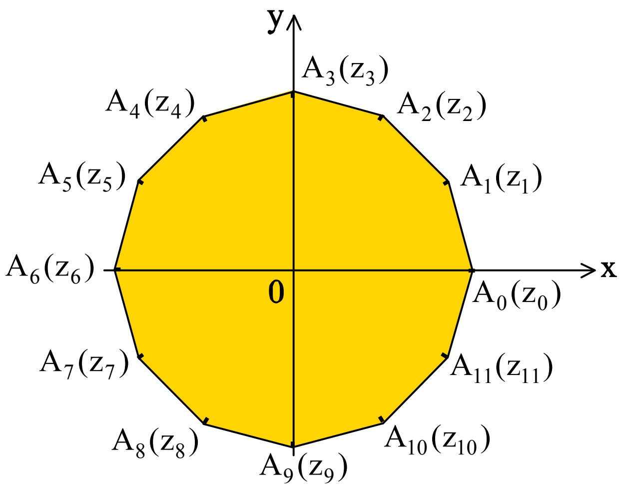
α) i) Show that $f(a) + f(b) > 2f\left(\frac{a+b}{2}\right)$ **2**

ii) Show that $f(a) + f(b) + f(c) + f(d) > 4f\left(\frac{a+b+c+d}{4}\right)$ **2**

β) Consider the curve $y = (x+1)^{-2p}$ where p is a positive integer and by using part (α), or otherwise, show that **3**

$$\frac{1}{(n+2)^{2p}} + \frac{1}{(n+4)^{2p}} + \frac{1}{(n+6)^{2p}} + \frac{1}{(n+8)^{2p}} > \frac{4}{(n+5)^{2p}}$$

(b) The roots of the polynomial $z^{12} - k^{12} = 0$, where k is a real number are the vertices of the dodecagon shown on the diagram.

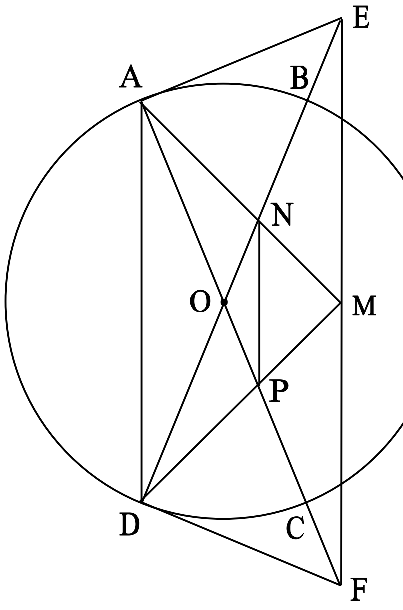


i) Find S the area of the dodecagon. **2**

ii) The point P represents the complex number w, where $|w| = |z|$. **2**
Show that $PA_0^2 + PA_1^2 + PA_2^2 + \ldots .. + PA_{10}^2 + PA_{11}^2 = 8S$
where S is the area of the dodecagon.

(c) AC and DB are diameters of the circle with centre O. The tangents at A and D meet DB and AC produced at E and F respectively.

The lines joining A and D to the midpoint M of EF intersect ED at N and DM at P as shown.



i) Show that triangle AMD is isosceles. **2**

ii) Show that the quadrilateral ANPD is an isosceles trapezium. **2**

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, n \neq -1;\ x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ \ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ \ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax dx, = \frac{1}{a}\tan ax,\ \ a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}dx = \ln(x + \sqrt{x^2 - a^2}),\ \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}dx = \ln(x + \sqrt{x^2 + a^2})$$

NOTE : $\ln x = \log_e x, \qquad x > 0$