Northern Beaches Secondary College

Manly Selective Campus

2011
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators and templates may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1- 8
- All questions are of equal value

Marks

Question 1 ***(Answer in a separate booklet)*** **15**

(a) $\int x \sec^2 (x^2)\, dx$ *(1)*

(b) $\int \dfrac{dx}{x(1 + x^2)}$ *(3)*

(c) Use the substitution $t = \tan \dfrac{x}{2}$ calculate $\displaystyle\int_0^{\frac{\pi}{2}} \dfrac{dx}{5 + 3 \sin x + 4 \cos x}$ *(4)*

(d) $\int \dfrac{dx}{\sqrt{x + 5}\,\sqrt{4 - x}}$ for $x < 4$ using the substitution $u^2 = 4 - x$. *(3)*

(e) $\int \sin (\ln x)\, dx$ using $u = \ln x$ *(4)*

Marks

Question 2 ***(Answer in a separate booklet)*** **15**

(a) Simplify:

(i) $(4 + 3i)^2$ *(1)*

(ii) $\dfrac{7 - 2i}{3 + i}$ *(1)*

(iii) $2 \text{ cis } \dfrac{\pi}{6} \times 3 \text{ cis } \dfrac{\pi}{3}$ *(1)*

(b) Find the roots of $z^5 - i = 0$ and sketch them on an argand diagram. *(3)*

(c) (i) In the same diagram, sketch the locus of both $|z - 2| = 2$ and $|z| = |z - 4i|$. *(2)*

(ii) What is the complex number represented by the point of intersection of these two loci? *(1)*

(d) Let $y = i(1 - i\sqrt{3})(\sqrt{3} + i)$

(i) Express y in cis form. *(2)*

(ii) Hence find $y^3 + y^{-3}$ in the form A + iB. *(2)*

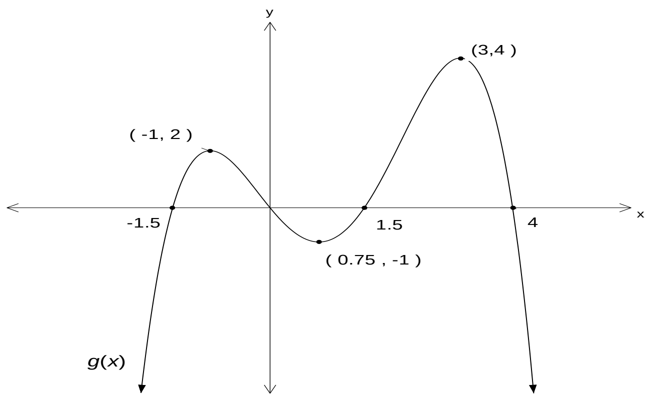
(e) Let z be a complex number of modulus 3 and ω be a complex number of modulus 1.

Show that $|z - \omega|^2 = 10 - \left(z\overline{\omega} + \bar{z}\omega\right)$ *(2)*

Marks

Question 3 ***(Answer in a separate booklet)*** **15**

(a) The diagram below shows the graph of $g(x)$.



Using this information, sketch the graphs:

(i) $k(x) = g(|x|)$ *(1)*

(ii) $t(x) = \dfrac{1}{g(x)}$ *(1)*

(iii) Graph $f(x)$ given $f(x) = \left(x - \dfrac{\pi}{2}\right)\cos x$ *(3)*

(b) Given the point $P\,(6\cos\theta\,, 2\sin\theta)$ lies on an ellipse, determine the following

(i) The eccentricity. *(1)*
(ii) Coordinates of the foci. *(1)*
(iii) Equations to the directrices. *(1)*
(iv) Determine the gradient to the ellipse when $\theta = \dfrac{2\pi}{3}$. *(2)*

(c) Given the polynomial $P(x) = 2x^3 + 3x^2 - x + 1$ has roots α, β and γ:

(i) Find the polynomial whose roots are α^2, β^2 and γ^2. *(2)*

(ii) Determine the value of $\alpha^3 + \beta^3 + \gamma^3$. *(3)*

Marks

Question 4 ***(Answer in a separate booklet)*** **15**

(a) Factorise the polynomial $P(x) = 3x^3 - 7x^2 + 8x - 2 = 0$ over $\mathbb{C}$. *(2)*

(b) Using the method of taking slices parallel to the x-axis, calculate the volume of the solid of revolution when the region bounded by the curve $y = \frac{1}{2}\sqrt{x-2}$, the x-axis and the line x = 6 is rotated around the line x = 6. *(4)*

(c) The area enclosed between the curves $y = (x-4)^2$ and $y = x + 2$ is rotated about the y-axis.

(i) Draw a diagram to show the area. *(1)*

(ii) By taking slices of the area parallel to the axis of rotation, show that the volume of the solid formed is given by $2\pi \int_2^7 9x^2 - x^3 - 14x.\, dx$ *(2)*

(iii) Find the volume of the solid formed. *(1)*

(d) A solid has as its base the region bounded by the curves $y = x$ and $x = 2y - \frac{y^2}{2}$.

Cross sections parallel to the x axis are equilateral triangles with a side in the base.

Determine the volume of this solid. *(5)*

Marks

Question 5 ***(Answer in a separate booklet)*** **15**

(a) (i) If $I_n = \int_0^1 (1 - x^2)^{\frac{n}{2}}\, dx$ where n is a positive integer, show that *(4)*

$$I_n = \frac{n}{(n+1)} I_{n-2}\ .$$

(ii) Hence evaluate I_5. *(2)*

(b) (i) The polynomial $f(x) = x^4 - 6x^3 + 13x^2 - ax - b$ has two double zeros α and β. Find the values of a and b. *(4)*

(ii) Hence determine, with full explanation, the equation of the line which touches the curve

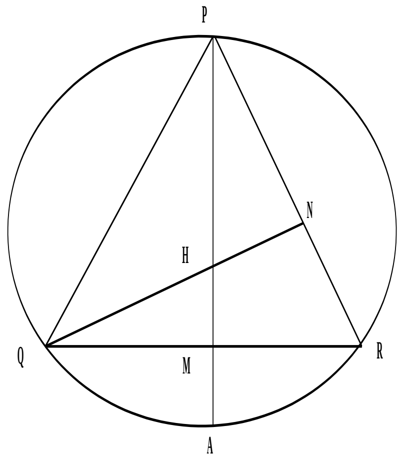
$$y = x^4 - 6x^3 + 13x^2$$

at two distinct points. *(1)*

(c) If $x + \frac{1}{x} = t$, find $x^6 + \frac{1}{x^6}$ in terms of t. *(4)*

Marks

Question 6 ***(Answer in a separate booklet)*** **15**



(a) The points P, Q and R lie on the circumference of a circle and form an acute angled triangle.

The altitudes PM and QN meet at H (which is <u>not</u> the centre of the circle)

(i) Prove that $R\hat{Q}A = R\hat{Q}N$ *(3)*

(ii) Prove that HM = MA *(1)*

(iii) Prove that RH produced meets PQ at right angles. *(2)*

(b) (i) Show that $a^2 + b^2 \geq 2ab$ where a and b are distinct positive real numbers. *(1)*

(ii) Hence show that $a^2 + b^2 + c^2 \geq ab + ac + bc$. *(1)*

(iii) Hence show that $\sin^2\alpha + \cos^2\alpha \geq \sin 2\alpha$. *(2)*

(iv) Hence show that $\sin^2\alpha + \cos^2\alpha + \tan^2\alpha \geq \sin\alpha - \cos\alpha + \sec\alpha + \frac{1}{2}\sin 2\alpha$. *(2)*

(c) One of the roots of the equation $kxe^{-x} - 4 = 0$ is a double root.
Find the value of k. *(3)*

Marks

Question 7 ***(Answer in a separate booklet)*** **15**

(a) Determine the equation to the tangent to the curve $x^3 + y^3 - 3x^2y^2 = 1$

at the point P (1,3). *(3)*

(b) A railway line has been constructed around a circular curve of radius 800metres, and is banked by raising the outer rail to a certain level above the inner rail.

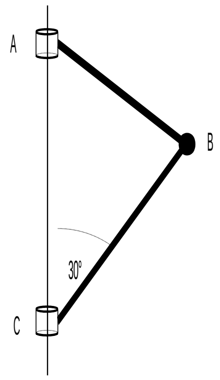
(i) When the train travels at 20m/s, the lateral thrust F, is on the outer rail. Show that

$F = m\left(\frac{1}{2}\cos\theta - g\sin\theta\right)$ where ϴ is the angle of inclination. *(2)*

(ii) When the train travels at 10m/s, the lateral thrust on the inner rail is the same as the lateral thrust on the outer rail at a speed of 20m/s.

a) Find the angle of the banking. *(2)*

b) Find the speed of the train when there is no lateral thrust exerted on the rails. Use $g = 9.8ms^{-2}$ *(2)*



(c) The above diagram shows a mass of 10 kilograms at B connected by light rods (at right angles) to sleeves A and C which revolve freely about the vertical axis AC but do not move vertically. The angle between the vertical axis AC and the light rod BC is 30°. The acceleration due to gravity is 9.8 m/sec^2.

(i) Given AC is 2 metres, show that the radius of the circular path of rotation of B is $\frac{\sqrt{3}}{2}$ metres. *(1)*

(ii) Find the tensions in the rods AB and BC when the mass makes 90 revolutions per minute about the vertical axis. *(5)*

Marks

Question 8 ***(Answer in a separate booklet)*** **15**

(a) Given the ellipse $\frac{x^2}{225}+\frac{y^2}{144}=1$, prove that the section of the tangent between the point of contact and its point of intersection with the directrix subtends a right angle with the corresponding focus.
(nb. The equation to the tangent to the ellipse is $\frac{x\,x_1}{a^2}+\frac{y\,y_1}{b^2}=1$ and does need to be proved.) *(3)*

(b) The hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with eccentricity e, has one focus S on the positive x-axis and the corresponding directrix d cuts the asymptotes to the hyperbola at points P and Q in the first and fourth quadrants respectively.

(i) Show that PS is perpendicular to the asymptote through P and that $PS=b$. *(3)*

(ii) A circle with centre S touches the asymptotes of the hyperbola. Deduce that the points of contact are the points P and Q. *(1)*

(iii) The circle with centre S which touches the asymptotes of the hyperbola cuts the hyperbola at points R and T. If $b=a$, show that RT is a diameter. *(2)*

(c) When $(1+ax)^5+(1+bx)^5$ in expanded in ascending powers of x, the expansion begins

$$2+30x+220x^2+\ldots$$

(i) Prove that $(a+b)=6$ and $(a^2+b^2)=22$ *(2)*

(ii) Deduce the value of ab. *(2)*

(iii) Determine the value of the coefficient of x^3. *(2)*

END OF EXAMINATION

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$