Northern Beaches Secondary College
Manly Selective Campus

2010
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators and templates may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1- 8
- All questions are of equal value

Marks

Question 1 *(Answer in a separate booklet)* ***(15)***

(a) Find $\int \frac{1}{\sqrt{4x^2 - 25}}\, dx$ *(2)*

(b) Find $\int \tan^4 x \sec^2 x\, dx$ *(2)*

(c) Find $\int_0^{\frac{\pi}{2}} x \sin x\, dx$ *(3)*

(d) Use the substitution $u = x$-2 to evaluate $\int_{\frac{3}{2}}^{\frac{5}{2}} \frac{1}{\sqrt{(x-1)(3-x)}}\, dx$ *(3)*

(e) (i) Write $\frac{3x+2}{x^2+5x+6}$ as a sum of partial fractions. *(2)*

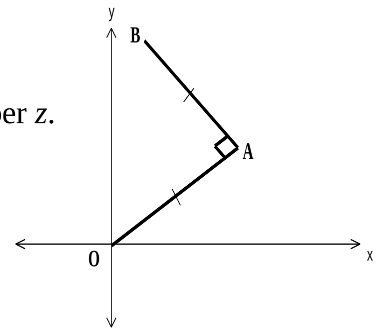
(ii) Hence evaluate $\int_0^2 \frac{3x+2}{x^2+5x+6}\, dx$ *(3)*

Marks

Question 2 *(Answer in a separate booklet)* ***(15)***

(a) Find the value of z^{10} in the form x + iy when $z = \sqrt{2} - \sqrt{2}\, i$ *(2)*

(b) The complex vector z is represented in the accompanying diagram by the point A.
The triangle OAB is a right angled isosceles triangle.



(i) Express the point B in terms of the complex number z. *(1)*

(ii) Let M be the midpoint of AB.
What complex number corresponds to M? *(1)*

(iii) Give an example of the vector C which would make a trapezium? *(1)*

(c) (i) Write down the value of i^6 . *(1)*

(ii) Hence or otherwise plot and label the sixth roots of -1 about the unit circle. *(2)*

(iii) Find the roots of $x^4 - x^2 + 1 = 0$ in modulus-argument form. *(2)*

(d) Given the locus of z is $|z - 2 - 2i| = 1$

(i) Sketch the locus of z on the argand diagram. *(1)*

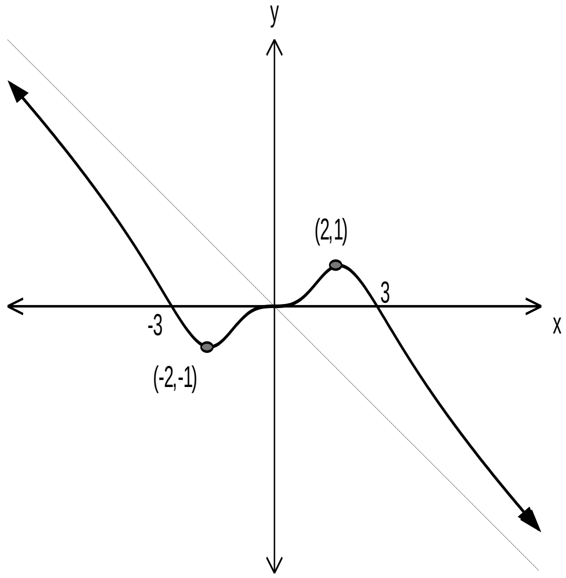
(ii) Find the maximum value of arg z. *(2)*

(iii) Find the maximum value of mod z. *(2)*

Marks

Question 3 *(Answer in a separate booklet)* ***(15)***

(a) The graph of $y = f(x)$ is shown. The line $y = -x$ is an oblique asymptote to the curve.



Use separate half page graphs, to sketch

(i) $f(-x)$ *(1)*

(ii) $f(|x|)$ *(1)*

(iii) $\dfrac{x}{f(x)}$ *(3)*

(b) Given that $1 + i$ is a zero of $P(x) = x^4 - x^3 - 2x^2 + 6x - 4$, factorise $P(x)$ fully over the field of the complex numbers. *(2)*

(c) The equation $x^3 + x^2 - 2x - 3 = 0$ has roots α, β and γ. Find the equation with roots:

(i) α^2, β^2 and γ^2 *(1)*

(ii) $\alpha^3\beta\gamma, \alpha\beta^3\gamma$ and $\alpha\beta\gamma^3$ *(2)*

(d) For the hyperbola $\dfrac{x^2}{16} - \dfrac{y^2}{25} = 1$,

(i) find the eccentricity. *(1)*

(ii) find the coordinates of the foci. *(1)*

(iii) find the equations of the directrices. *(1)*

(iv) find the equations of the asymptotes. *(1)*

(v) if P is on the hyperbola and S and S' are it's foci, then given PS=2, find PS'. *(1)*

Marks

Question 4 *(Answer in a separate booklet)* **(15)**

(a) Using the method of cylindrical shells, find the volume generated by revolving the region bounded by $y = \log_e x$, the x-axis and the interval $1 \leq x \leq e$ about the y axis. *(4)*

(b) The ellipse $\frac{(x-4)^2}{9} + \frac{y^2}{4} = 1$ is rotated about the y axis to form a donut shape.

(i) By taking slices perpendicular to the axis of rotation, show that the volume of a slice is $8\pi\sqrt{36 - 9y^2}\ \delta y$ *(2)*

(ii) Find the volume of the solid *(2)*

(c) Calculate the modulus and argument of the sum of the roots of the equation *(2)*

$$(3 + 4i)z^2 + (2 - i)z + (8 - 2i) = 0$$

(d) PQ is a chord of a rectangular hyperbola $xy = c^2$

(i) Show that PQ has equation $x + pqy = c(p+q)$ where P and Q have parameters p and q respectively. *(2)*

(ii) If PQ has a constant length k^2, show that

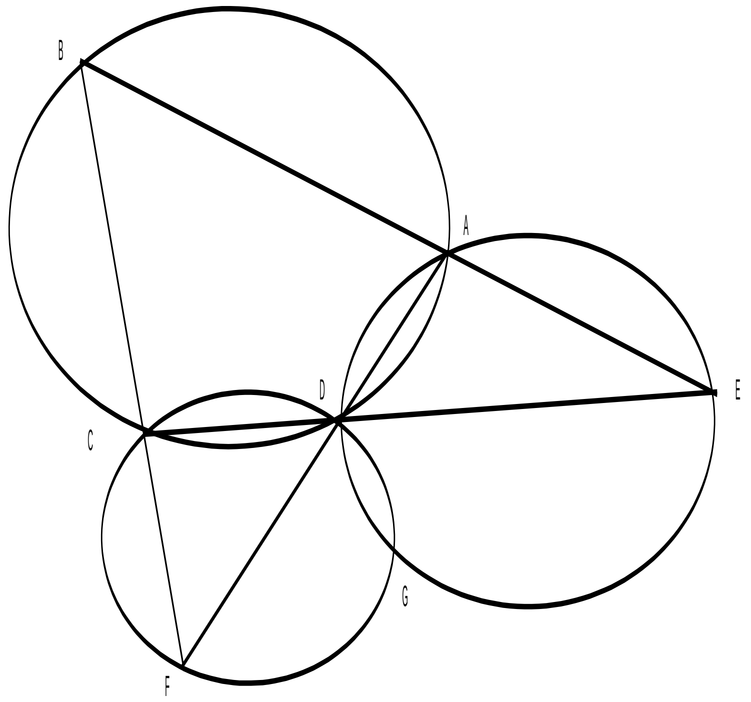
$$c^2[(p+q)^2 - 4pq]\,(p^2q^2 + 1) = k^4p^2q^2$$

and find the locus of R, the midpoint of PQ, in Cartesian form. *(3)*

Marks

Question 5 *(Answer in a separate booklet)* ***(15)***

(a) ABCD is a cyclic quadrilateral. BA and CD are both produced and intersect at E. BC and AD produced intersect at F. The circles EAD and FCD intersect at G as well as at D. Prove the points E, G and F are collinear. *(4)*



(b) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$ for integers n, $n \geq 0$

(i) Show that $I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \, dx$ for $n \geq 2$. *(2)*

(ii) Deduce that $I_n = \frac{n-1}{n} I_{n-2}$ for $n \geq 2$. *(2)*

(iii) Evaluate I_4 *(2)*

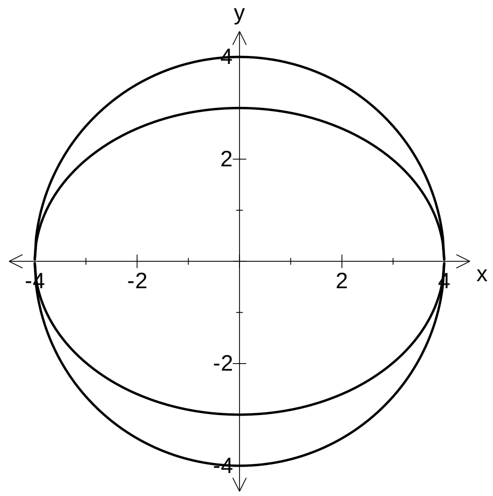
(c) (i) If α is a multiple root of the polynomial equation $P(x) = 0$, prove that $P'(\alpha) = 0$ *(2)*

(ii) Find all roots of the equation $18x^3 + 3x^2 - 28x + 12 = 0$ if two of the roots are equal. *(3)*

Marks

Question 6** (Answer in a separate booklet)* ***(15)

(a) The diagram shows an ellipse with equation $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and the larger of its auxiliary circles. The coordinates of a point P on the ellipse are $(4\cos\theta, 3\sin\theta)$ where $\theta \neq 0$ or π.



A straight line l parallel to the y axis intersects the x axis at N and the ellipse and the auxiliary circle at the points P and Q respectively.

(i) Find the equations of the tangent to the ellipse at P and to the auxiliary circle at Q. *(4)*

(ii) The tangents at P and Q intersect at point R. Show that R lies on the x axis. *(2)*

(iii) Prove that ON*OR is independent of the positions of P and Q. *(1)*

(b) On Tuesday morning at 5 am, a truck crashes into a harbour. The rescue team and their equipment can only work effectively when the depth of water is no more than 7 m. At low tide, the depth of the water is 5 metres and at high tide, the depth is 10 metres. Low tide occurs at 4 am and high tide at 10.15 am. Assume the movement of the tide is simple harmonic motion.

(i) Find the period and amplitude of the motion. *(2)*

(ii) If the deadline for the rescue operation is 6 pm on Wednesday evening, find the periods of time between 5 am and 6 pm during which the rescue team can work effectively. *(4)*

(c) Given a real polynomial Q(x) show that if α is a root of Q(x)-x=0, then α is also a root of Q(Q(x))-x=0. *(2)*

Marks

Question 7 *(Answer in a separate booklet)* ***(15)***

(a) For the curve $x^2y^2 - x^2 + y^2 = 0$

(i) state any x and y-intercepts. *(1)*

(ii) demonstrate why $|y| < 1$. *(1)*

(iii) demonstrate why $|y| \leq |x|$. *(1)*

(iv) use implicit differentiation to show $\dfrac{dy}{dx} = \dfrac{x(1-y^2)}{y(1+x^2)}$. *(2)*

(v) state the co-ordinates of any critical points (where the derivative is undefined). *(1)*

(vi) hence, explain why the curve has no stationary points. *(1)*

(vii) state the horizontal asymptote(s). *(1)*

(viii) sketch the curve. *(1)*

(b) By expanding $(\cos\theta + i\sin\theta)^3$ it can be shown that

$$\cot 3\theta = \frac{t^3 - 3t}{3t^2 - 1} \text{ where } t = \cot\theta$$

(i) solve $\cot 3\theta = -1$ for $0 \leq \theta \leq 2\pi$ *(2)*

(ii) Hence show that $\cot\dfrac{\pi}{12} \cdot \cot\dfrac{5\pi}{12} \cdot \cot\dfrac{9\pi}{12} = -1$ *(2)*

(iii) Write down a cubic equation with roots $\tan\dfrac{\pi}{120}$, $\tan\dfrac{5\pi}{12}$ and $\tan\dfrac{9\pi}{12}$. *(2)*

(Express your answer as a polynomial equation with positive integer coefficients).

Marks

Question 8 *(Answer in a separate booklet)* ***(15)***

(a) The series $\frac{1}{2} + \frac{8}{4} + \frac{27}{8} + \,..\, = \sum_{n=1}^{\infty} \frac{n^3}{2^n}$ is not geometric and, as such, it is not a routine matter to decide whether or not it converges to a finite sum. Let $y_n = \frac{n^3}{2^n}$

(i) Show that $\frac{y_n}{y_{n-1}} = \frac{1}{2} \times \left(\frac{n}{n-1}\right)^3$ and hence show that this ratio is greater than 1 when $2 \le n \le 4$ but less than 1 when $n \ge 5$. *(3)*

(ii) Show that $\frac{y_n}{y_{n-1}} \le 0.98$ for $n \ge 5$. *(2)*

(iii) Given that $y_4 = 4$, deduce that $y_n \le 4 \times (0.98)^{n-4}$ for $n \ge 4$ and write down the value of $\lim_{n \to \infty} y_n$. *(2)*

(b) Show that the derivative of the function $y = x^x$ for $x > 0$ is $(\log_e x + 1)\, x^x$. *(2)*

(c) Find all solutions $(0 \le \theta < \pi)$ in radians of the equation $\frac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta} = \frac{3}{4}$. *(3)*

(d) A triangle ABC is right-angled at A and it has sides of lengths a, b and c units (the side opposite angle A is a etc). A circle of radius r units is drawn so that the sides of the triangle are tangents to the inscribed circle.

Prove that $r = \frac{1}{2}(c + b - a)$. *(3)*

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$