BOARD OF SENIOR SCHOOL STUDIES

MATHEMATICS

FIRST LEVEL

Syllabus and Notes - approved by the Board 3.3.65.

Corrigenda to and amplification of Notes on Geometric Algebra and Matrices.

(Approved by the Board on 4 August 1967)

The list below sets out the correct form of a phrase or expression that has been misprinted. Misprints which do not obscure the sense are not included in the list.

page line

28 -2 of given magnitude α, the

29 -6 $\mathcal{T}_{\alpha+2\pi} = \mathcal{T}_{\alpha}, \quad (\mathcal{T}_{\alpha})^n = \mathcal{T}_{n\alpha}$

29 -2 operation $\mathcal{S}_a$ of reflection

30 8 $2.\ \mathcal{S}_b\mathcal{S}_a = \mathcal{T}_{2\angle^* ab}$

30 10 $0 \leq \angle^* ab < \pi$

30 11 $\mathcal{S}_a\mathcal{S}_b = \mathcal{T}_{2\angle^* ba} = \mathcal{T}_{2(\pi - \angle^* ab)} = \mathcal{T}_{-2\angle^* ab}$

$$= \mathcal{T}^{-1}_{2\angle^* ab} = (\mathcal{S}_b\mathcal{S}_a)^{-1}$$

30 -12 Proof of relation in this line

$$\mathcal{S}_c\mathcal{S}_b\mathcal{S}_a = \mathcal{S}_d \Rightarrow (\mathcal{S}_c\mathcal{S}_b\mathcal{S}_a)^{-1} = \mathcal{S}_d^{-1} = \mathcal{S}_d$$

$$\Rightarrow \mathcal{S}_a\mathcal{S}_b\mathcal{S}_c = \mathcal{S}_d = \mathcal{S}_c\mathcal{S}_b\mathcal{S}_a$$

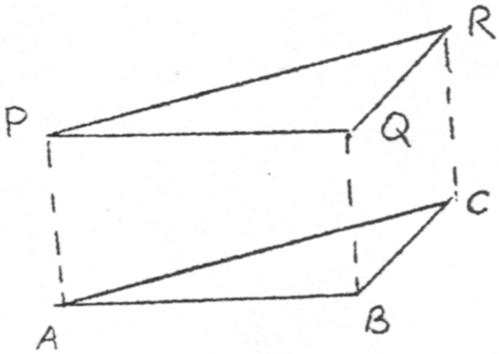
30 -7,6 Better written as $h \perp a$ and $H \in h$ and $H' \in \mathcal{S}_a H \Rightarrow H' \in h$

Conversely, $H' = \mathcal{S}_a H \Rightarrow HH' \perp a$

31 3-6 These are statements about operators. The form in which the theorems have to be proved is

1. $\mathcal{D}_{BC}\mathcal{D}_{AB}P = \mathcal{D}_{AC}P$ for all P.

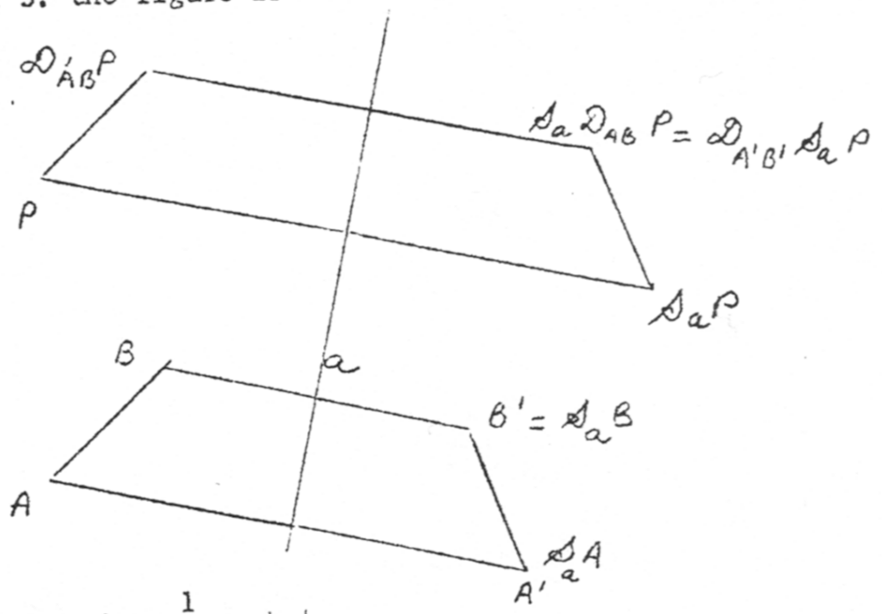
and the figure for the proof is



The essence of the proof is

$AB \parallel PQ$ and $BC \parallel QR$ and $AC \parallel PQ$ and $AP \parallel BQ \Rightarrow AP \parallel CR$

For 3. the figure is



34 15 where $\theta = \frac{1}{2}\pi + \phi$

35 1 $(k_1 + k_2)\, \underline{\underline{r}} = [\, \underline{\underline{r}}_1 \quad \underline{\underline{r}}_2 \,]\, \underline{\underline{k}}$

35 9 to 11 if $\mathscr{M} P_i = P_i'$ and $P_3 \varepsilon P_1P_2$, we have

$$P_3' \varepsilon P_1' P_2'$$

and $$\frac{{}^*P_1 P_3}{{}^*P_1 P_2} = \frac{{}^*P_1' P_3'}{{}^*P_1' P_2'}$$

36 16,17 $$\underline{M}\begin{bmatrix} x_o \\ y_o \end{bmatrix} = \begin{bmatrix} ax_o + by_o \\ cx_o + dy_o \end{bmatrix} = \begin{bmatrix} ax_o + by_o \\ \dfrac{c(ax_o + by_o)}{a} \end{bmatrix}$$

41 -12 the transformation from (X,Y) to (X', Y') is

43 -10,-9 so that, if $h \neq 0$

$p = 0$ and $s = 0$ and $h \neq 0 \Rightarrow q = 0$ and $r = 0$

-4 $h \neq 0$

-1 $k \neq 0$

44 4 $bb'z^2 - (bc + b'c' + h^2)yz + cc'y^2 = 0$

44 7 $bc + b'c' + h^2 = 0$

45 6 Always singular if and only if $b' = c' = 0$

47 4 the element in position 12 on the matrix $T_\theta^T C T_\theta$

47 -12 $\lambda\alpha$ corresponds to $\rho \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$

48 -15 S, T, $\underline{1}$ for $\mathscr{S}, \mathscr{T}, \mathscr{I}$

-13 S: reflection in an altitude through the point in the plane originally occupied by the vertex R.

50 -10 to -7 Add after line -7 the following explanation:

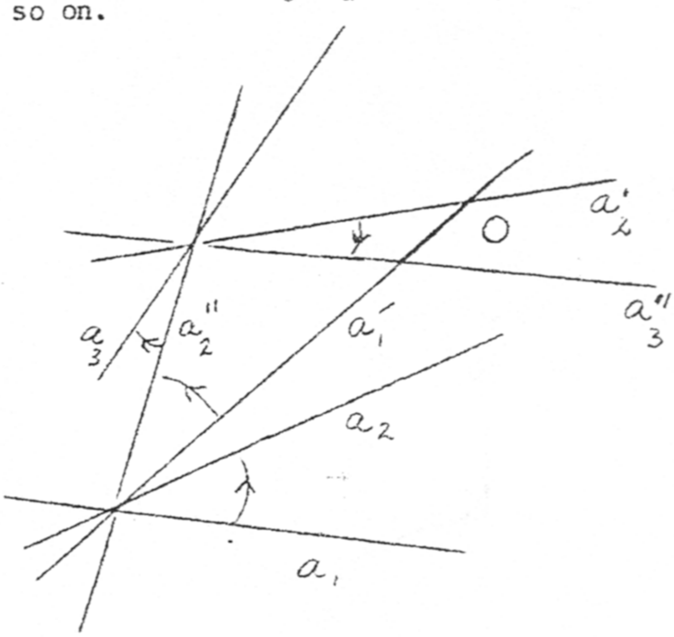
To effect these reductions, replace successive pairs of reflections by pairs, the axes of the first members of which pass through O. (The term "axis" of reflection has not so far been used, but can be introduced earlier with advantage.) Let $a_1, \ldots, a_n$ be the axes of reflections $\mathcal{S}_1, \ldots, \mathcal{S}_n$.

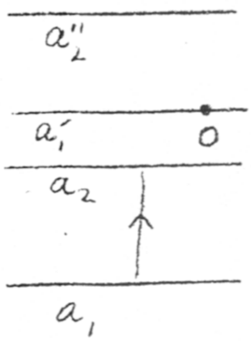
Find a_1', a_2'' (axes of $\mathcal{S}_1', \mathcal{S}_2''$) such that

$\mathcal{S}_2''\mathcal{S}_1' = \mathcal{S}_2\mathcal{S}_1$ and a_1 passes through O.

(There are two cases to discuss $a_1 \not\parallel a_2$ and $a_1 \parallel a_2$).

Next replace $\mathcal{S}_3\mathcal{S}_2''$ by $\mathcal{S}_3''\mathcal{S}_2'$ with a_2' through O and so on.





In this way $\mathcal{S}_n\mathcal{S}_{n-1} \cdots \mathcal{S}_2\mathcal{S}_1$ is replaced by

$\mathcal{S}''_n\mathcal{S}'_{n-1} \cdots \mathcal{S}_2'\mathcal{S}_1'$, with the axes $a_1', a_2', \ldots, a'_{n-1}$ all passing through O, but a''_n not in general doing so.

Then $\mathcal{S}'_{n-1} \cdots \mathcal{S}_2'\mathcal{S}_1' = \begin{cases} \mathcal{S}_0 & \text{if } n-1 \text{ is odd} \\ \mathcal{S}_0'\mathcal{S}_0 \text{ or } \mathcal{I} & \text{if } n-1 \text{ is even.} \end{cases}$

(for axes a_0, or a_0' and a_0 through O completely defined by $a_1, a_2, \ldots, a_n$)

51 13 $$\underline{\underline{N}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

54 -1 $$(\underline{\underline{1}} - \underline{\underline{M}})^n$$

BOARD OF SENIOR SCHOOL STUDIES

MATHEMATICS

First Level Syllabus

1. THEORETICAL ARITHMETIC

(a) The Integers or Whole Numbers

Review of the fundamental properties of the set of integers. Factorisation and divisibility. Primes and composite numbers. The division transformation. Highest common factor. Euclid's algorithm and deductions from it. The unique factorisation into primes.

(b) The Rational Numbers

Review of the origin of the rational numbers. Comparison of the set of rationals with the set of integers. Decimal representation of rationals.

(c) The Real Numbers

Set of all decimals as an ordered set in which arithmetical operations may be defined so as to give this set the same essential structure as that of the set of rationals. Comparison of the set of real numbers with the set of rational numbers. The monotonic principle of convergence. The definition of a^x for $a > 0$.

(d) Sequences and Series

Notion of a limit in connection with sequences. Convergence of series. Comparison test for absolute convergence. Simple examples of conditionally convergent series. Demonstration of the limits:

for $o < a < 1$, $p \leq o$, $n^p a^n \to o$ as $n \to \infty$

for $x > 1$, $\dfrac{x^n}{n!} \to o$ as $n \to \infty$

(e) Complex Numbers

Origin of the idea in connection with the solution of quadratic equations. Exploration of the possibilities by introduction of the imaginary i, $i^2 = -1$, and recognition of the formal theory as a calculus of ordered pairs of real numbers. Modulus, argument, conjugate. Geometric representation of addition and multiplication of complex numbers in the Argand diagram. The relations

$|z_1 + z_2| \leq |z_1| + |z_2|$, $|z_1 z_2| = |z_1| \cdot |z_2|$,

$\arg z_1 z_2 = \arg z_1 + \arg z_2$

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2. ALGEBRA

(a) Polynomials

The general notion. The "indeterminate", coefficients, degree. The influence of the coefficient set on the structure of the theory. Addition and multiplication of polynomials. Factorisation and divisibility. Prime and composite polynomials. Highest common factor. The division transformation. Euclid's algorithm and deductions from it. The unique factorisation into primes.

(b) Polynomials as Functions

Polynomial equations. Roots. The remainder and factor theorems. Relation between the number of roots and the degree. The identity of two polynomials. Relations between the roots and the coefficients in case of a polynomial which is completely reducible to linear factors.

(c) Taylor's Theorem for Polynomials

The binomial theorem for a positive integer index. The derived polynomial $Df = f'$. Taylor's Theorem. Multiple roots.

(d) Rational Functions

Partial Fractions.

3. CALCULUS.

(a) The Function Concept

The notion of related variables. Dependent and independent variables. Functions as mappings or correspondences. The functional notation. Graphical representation. Inverse functions.

(b) The Continuous Function of a Real Variable

Meaning of continuity at a point and in an interval. Statement of the fundamental properties without proof.

(c) Tangents to Curves

Gradient or slope. Calculation of the gradient as a limit. Differentiation. The derived function. The significance of the sign of the derived function. Maxima and Minima. Rolle's theorem. The mean value theorem. Differentiation of combinations of functions and of composite functions. Differentiation of rational functions and of simple irrational functions. The relation

$$\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$$

in connection with inverse functions. Geometric meaning on the graph. Calculation of derivative of functions defined implicitly in cases which do not involve partial differentiation. Second derivatives, inflexions and curve tracing.

(d) The Problem of Areas

Figures bounded by curved lines. The definite integral defined as the limit of a sum. Formal properties of the definite integral. Indefinite integrals and the fundamental theorem of the calculus. Calculation of areas and volumes.

(e) The Exponential and Logarithmic Functions

The theory of these functions should be derived from one or other of the differential equations

$$\frac{dy}{dx} = y \ , \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{x}$$

The definition of e. Derivation of the exponential and logarithmic series. Calculation of e and of $\log_e 2$.

(f) Arcs of Curves; the Trigonometric Functions

Calculation of length of arc. Length of circular arc. Measure of an angle. The radian. Definition of π. The trigonometric or circular functions defined by reference to the unit circle. Graphs of these functions. Differentiation of the trigonometric functions. The addition formulae and allied formulae.

Trigonometric equations. Series for sin x and cos x. The inverse circular functions and their "principal" values. Differentiation of $\sin^{-1}x$, $\cos^{-1}x$ and $\tan^{-1}x$. Series for $\tan^{-1}x$. Calculation of π.

(g) Integration

The table of standard integrals. Indefinite integration. Integration by parts, and use of change of variable. Integration of simple rational functions and of simple irrational functions involving the square root of a quadratic.

4. PLANE GEOMETRY. GEOMETRIC ALGEBRA. MATRICES. GEOMETRY IN THREE DIMENSIONS.

(a) Elementary Plane Analytical Geometry

Cartesian co- ordinates. Distance formula, section formula. Forms of equation of the straight line including parametric forms. Distance of a point from a line. Point of intersection of two lines. Angle between two lines.

3 x 3 determinants. Area of a triangle with given vertices. Condition of concurrence of three lines.

Simple geometry of the parabola.

(b) Linear Transformations in the Plane. Introduction of Matrix Algebra.

Rotations, reflections, displacements as geometric operators. Groups of operators. Brief treatment of groups in general. Cartesian form of rotations etc. 2 x 2 matrices as operators.

2 x 2 matrices as complex numbers.

General form and properties of 2 x 2 matrices. Formal rules for combination of 2 x 2 matrices under multiplication and addition. Singular 2 x 2 matrices and divisors of the zero matrix.

Re-interpretation of matrix relations as changes of co-ordinates. Combination of change of co-ordinates with linear transformations; similarity.

Characteristic function of 2 x 2 matrix; eigenvalues and eigenvectors.
Cayley-Hamilton theorem. Invariance under similarity.

Application to the reduction of the general quadratic form in two variables (conic). Change or origin, significance $\Delta = 0$, $C = 0$. Lengths and directions of principal axes from eigenvalues and eigenvectors.

Relation of curves represented by vanishing of a quadratic form to conics defined by focus-directrix property.

(c) Analytical Geometry in Three Dimensions

(The content of this is largely included in the IIF course and may be presented either in traditional form or by introducing 3 x 3 matrices and expressing relations in matrix form.)

Planes and lines in 3 dimensions. Angle between plane and line. The normal to a plane. Skew lines and the angle between them.

Cartesian co-ordinates in 3 dimensions. Distance between two points. Section formulae. Direction angles and direction cosines. Determination of the angle between two lines given their direction cosines. Condition for perpendicularity.

The general equation of a plane and the perpendicular form. Conditions for two planes to be parallel and perpendicular. Distance from a point to a plane. Intersection of three planes and relation to algebraic solution of three linear equations.

The equation of a sphere and the general notion of the equation of a surface. Three dimensional interpretation of relations which involve only two co-ordinates.

5. ELEMENTARY DYNAMICS OF A PARTICLE.

Rectilinear motion of a particle described by a functional relation $x = f(t)$. Velocity and acceleration as differential coefficients. Kinematical formulae for uniformly accelerated motion. Newton's Laws of motion - first and second laws. Differential equations of motion in one dimension. Momentum, work and energy. Kinetic and potential energy. Equation of energy. Resisted motion, terminal velocity. Simple harmonic motion. Projectiles. Motion in two dimensions under gravity.

6. THEORY OF PROBABILITY.

(a) Statistical regularity. Random experiments. Relative frequency as an empirical measure of probability.

(b) Random experiments with a finite number of possible outcomes. Simple events, composite events. Probability of an event. Mutually exclusive events; the opposite (complementary) event. The algebra of events. Theorem of total probability.

(c) Two stage random experiments. Independent events. The product rule.

(d) Systematic enumeration in a finite sample space leading to the definitions of ${}^nP_r$ and ${}^nC_r$.

(e) Binomial probabilities and the binomial distribution.

(f) The notion of a random variable, illustrated mainly in connection with the binomial distribution. The expected value of a random variable. The expected value of the binomial variable.

BOARD OF SENIOR SCHOOL STUDIES

MATHEMATICS

First Level Syllabus

NOTES

The First Level Syllabus is arranged as six main themes:

1. Theoretical Arithmetic
2. Algebra
3. Calculus
4. Geometric Algebra
5. Elementary Dynamics
6. Theory of probability

The first four themes are intended to be developed side by side while the fifth theme can be added as soon as the Calculus is strong enough, and the sixth may be treated at almost any time.

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1. THEORETICAL ARITHMETIC

(a) The Integers or Whole Numbers

The fundamental properties will be known from the work of the first four years, but may be summarily reviewed. The essential facts are:

(i) the set of integers is ordered,

(ii) two binary operations, addition and multiplication are defined within the set and these obey "algebraic" laws which have been fully discussed in earlier work,

(iii) there are relations between the properties (i) and (ii) and these may be called the "laws for inequalities". There are two principal relations and these are not independent. They are:

a, b, c being integers, and $a < b$ then $a + c < b + c$;

and

a, b, c being integers, $a < b$ and $c > o$ then $ac < bc$.

These are the basic facts for work with inequalities.

Divisibility among the integers is defined as follows:

Given a, $b \neq o$, if there is an integer c such that $a = bc$ we say " b divides a" and indicate this by $b \mid a$. "b does not divide a" is indicated by $b \nmid a$. We say also "b is a factor of a"; and also c is a factor of a. The two theorems.

$$b \mid a, \ \& \ b \mid a' \longrightarrow b \mid a \ (a + a'),$$

and

$$b \mid a, \ \& \ c \mid b \longrightarrow c \mid a,$$

are consequences of the algebraic laws. For much of the discussion it is convenient to restrict attention to the positive integers and their positive factors. If we do this the factorisation $a = bc$ implies $1 \leq b \leq a$ and $1 \leq c \leq a$. So any positive a can have only a finite number of positive factors. If $a > 1$ it has the two factors 1 and a; and there are numbers a which have no other factors.

Such numbers are prime; and numbers which are not prime are called composite. A composite number a (>1) has a divisor d with $1 < d < a$. It follows that any number >1, if not prime, can be expressed as a product of a finite number of prime factors.

If $\{p_1, \ldots, p_n\}$ is a finite set of primes and we form the number $N = 1 + p_1 p_2 \ldots p_n$ then we see $p_1 \nmid N$; for if $p_1 \mid N$ we infer $p_1 \mid 1$, and so $1 = ap_1 > a \geq 1$, $1 > 1$. Similarly $p_i \nmid N$, $i = 1, \ldots, n$. Thus N has a prime divisor $p \leq N$, which must be different from each of the p_i. Thus we infer that there are infinitely many primes.

If $a > 1$ is composite we may write $a = bc$ with $b \geq c > 1$. Hence $a = bc \geq c^2$. Thus a has a divisor c whose square does not exceed a. In particular any composite a will have a prime divisor p with $p^2 \leq a$. To test whether a given number $a > 1$ is prime it is sufficient to test it for divisibility by all the primes p for which $p^2 \leq a$.

Given $b > o$, there is a unique largest integer q such that $qb \leq a$. Then $qb + b = (q + 1) b > a$. If we set $a = qb + r$ then r is uniquely determined and $o \leq r < b$. We call q the "quotient" and r the remainder when a is divided by b. The relation.

$$a = qb + r, \quad 0 \leq r < b$$

is called the division transformation.

The highest common factor of two integers a, b is the largest positive number h such that $h \mid a$, $h \mid b$. If $h = 1$ we say a, b are relatively prime. If now $a = qb + r$, $0 \leq r < b$ we infer $h \mid r$.

Given two positive integers $a_0 \geq a_1$ we can form the Euclid algorithm

$$\begin{array}{lll}
a_0 = q_1 a_1 + a_2, & & 0 \leq a_2 < a_1 \\
a_1 = q_2 a_2 + a_3, & & 0 \leq a_3 < a_2 \\
\ldots \quad \ldots & \ldots & \ldots \quad \ldots \\
a_{r-2} = q_{r-1} a_{r-1} + a_r, & & 0 \leq a_r < a_{r-1} \\
a_{r-1} = q_r a_r & &
\end{array}$$

The remainders decrease and the process ends after r steps when the last remainder is zero. Then, by a simple argument, $a_r = h$, the H.C.F. of a_0, a_1.

Now working "up" the algorithm we find

$$h = a_r = A_o a_o + A_1 a_1$$

where A_o and A_1 are integers. Changing the notation we see that if a, b are any positive integers there are integers

A, B such that

$$Aa + Bb = h = \text{H.C.F. of } a,\ b.$$

In this statement the restriction to positive a, b is obviously unnecessary. When a, b are relatively prime, h = 1 and we have the relation

$$Aa + Bb = 1$$

Conversely this relation implies that a, b are relatively prime. If now $a \mid bc$ we have

$$Aac + Bbc = c$$

but a divides both the terms on the left, so $a \mid c$. If a divides a product and is prime to one factor then it divides the other. Thus if the prime p divides the product of primes $p_1 \ldots p_n$ then we must have $p = p_i$ for some i. For if not we derive successively

$$p \mid p_2 \ldots p_n, \quad p \mid p_3 \ldots p_n, \quad \ldots, \quad p \mid p_n$$

and this is a contradiction. The fundamental theorem on the uniqueness of the prime factorisation of any number a now follows easily.

(b) The Rational Numbers

The fundamental properties concerning the order and the arithmetic operations in the set of rational number will be known. They are precisely as detailed above for the set of integers with one additional property, namely, given any rational number $a \neq 0$ there is one and only one rational number b such that ab = 1. This b is denoted by a^{-1}.

Given any two rationals a, $b \neq 0$ there is just one rational c such that a = bc. Thus if we take $c = b^{-1}a$ then $bc = b(b^{-1}a) = 1a = a$. One consequence of this fact is that there is no theory of divisibility among the set of rationals in any way similar to what we have considered for the integers.

Between any two rationals there is another rational and hence infinitely many rationals. Thus if $a < b$ we have

$$2a = a + a < a + b < b + b = 2b.$$

$$\therefore \quad a < \frac{a+b}{2} < b.$$

It should be remembered that the rationals are constructed from the integers as ordered pairs of integers on which the operations of addition and multiplication are defined. Mathematicians do this formally, but it should be noted that school children do exactly the same thing, only less formally.

Thus for instance the "fraction" 2/3 is literally an ordered pair of integers written vertically and separated by a bar, instead of horizontally and separated by a comma - and 2/3 is identified with 4/6; etc. Anyone who knows the formal construction will see that it is not much ahead of the school construction, and that the biggest difference is in the mental attitude to the process.

The decimal representation of the rational numbers is also familiar since primary school. This should be reviewed noting especially that the decimals obtained either terminate or recur. In the conversion of rationals to their decimal representation such an infinite decimal as

$$2.20200200020000 2 \ldots$$

which neither terminates nor recurs does not arise.

(c) The Real Numbers

The need for a further extension of the set of numbers beyond the rationals may be illustrated by the simplest arithmetic problems. We may seek, among the rationals, a number x which satisfies any of the equations such as

$$x^2 = 2, x^2 = 3, x^3 = 4.$$

But we will seek in vain! It is easy to prove that a rational number x does not satisfy any one of these equations. In earlier work we have been accustomed to write, as solutions of these equations, $x = \sqrt{2}$, $x = \sqrt{3}$, $x = \sqrt[3]{4}$. If there is any sense in this formalism we have not yet explained it. Our purpose should be to construct, if possible, a number system which will contain all the numbers (i.e., the rationals) that we have already, and others besides which may satisfy these equations. This extended number system should have the essential properties of the system of rationals as we have explained them above.

There are in fact several extensions of the set of rationals which have been studied and which are of great interest. If, for instance, we were only concerned to have a number system in which all polynomial equations have solutions then we would be satisfied to introduce the so-called "algebraic" numbers. If, however, we wish to establish the customary limit processes involved in, say, the calculus we require a much wider extension.

The germ of the appropriate extension has already presented itself in primary school work, in the decimal representation used there. The real numbers may be introduced as the set of all decimals - not merely those which terminate or recur. It can be easily explained how this set is ordered; how addition and multiplication is defined within the set; and how, when these definitions are properly given, (and the proper way is "nearly obvious") the set of real numbers has all the properties (i), (ii), (iii) which we have picked out above

as the essential properties of the set of rationals. This is precisely our justification for calling the set of decimals, with the structure we have imposed on it, a set of numbers; and the word real is used merely as a name. The numbers of this set which do not terminate or recur are the irrational numbers.

But the set of real numbers, now introduced, has further important properties which are not possessed by the set of rationals; naturally it is these new properties which gives this set of reals their importance. A fundamental property of this kind (from which all the other important ones follow) is what we call the monotonic principle of convergence. While this is easy enough to prove, such a formal proof is not required at this stage. What is required is that the principle should be accurately stated, discussed and explained sufficiently so that pupils feel reasonably convinced of its accuracy, and so that they can use it and argue from it accurately. If necessary the principle could be set up as a postulate, to be accepted for the present, and to be examined more fully at a later stage. There would be no violation of logic in this procedure. It is easily shown by examples that the principle does not hold within the set of rational numbers.

Now we can prove that each of the equations $x^2 = 2$, etc., quoted above has a solution among the real numbers; the solutions are of course irrational and as a matter of notation they are denoted by $\sqrt{2}$, $\sqrt{3}$ etc. By exactly the same argument we can now show that any equation like

$$x^q = a^p$$

where $a > 0$ is real and p, q are integers has exactly one positive solution. This is denoted by $a^{p/q}$. Here, and for the first time, we have a satisfactory definition of the meaning of fractional exponents. The usual "laws of indices" for such exponents now follow by the usual arguments. For example for positive integers p, q, r, s, if

$$x = a^{p/q}, \qquad y = a^{r/s}$$

we have

$$(xy)^{qs} = x^{qs}\, y^{qs} = a^{ps}\, a^{rq} = a^{ps+rq}$$

So $xy = a^{(ps+rq)/qs} = a^{p/q + r/s}$

Hence $a^{p/q} . a^{r/s} = a^{p/q + r/s}$

(d) Sequences and Series - These considerations lead naturally to the discussion of sequences and series, to the introduction of the notion of the limit of a sequence, and to the convergence of series.

The particular limits

when $p \geqq o$, $O < a < 1$, $n^p a^n \longrightarrow 0$ as $n \longrightarrow \infty$;

when $x > 1$, $\frac{x}{n!} \rightarrow o$ as $n \rightarrow \infty$

should be established.

Here are simple proofs, for $o < b < 1$,

$$b^n = \frac{nb^n}{n} < \frac{1}{n} \quad (b + b^2 + \ldots + b^n) \quad \frac{b}{n(1-b)}$$

Raise to power $1 + p$ and set $a = b^{1+p}$. We get

$$n^p a^p < \frac{1}{n} \left(\frac{b}{1-b}\right)^{1+p} \rightarrow 0 \text{ as } n \rightarrow \infty$$

For $n > r + 1 > 2x$,

$$n! > n(n-1)\ldots(r+1) > (2x)^{n-r} = \frac{2^n x^n}{(2x)^r}$$

$$\frac{x^n}{n!} < \frac{(2x)^r}{2^n} \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

The comparison test for convergence of a series of positive terms follows immediately from the monotonic principle. The convergence of the infinite G.P. and the series $\Sigma \frac{1}{n^p}$ should be established.

Absolute convergence of an infinite series should be defined and it should be shown that absolute convergence implies convergence. The possibility of convergence without absolute convergence should be illustrated by the example

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots\ldots$$

(e) Complex Numbers. There remain simple algebraic questions which have no answer within the set of real numbers. There is no real number x which satisfies the equation $x^2 + 1 = 0$. We may wish to extend the number system further so that in the new set of numbers equations of this kind have solutions. Here also the extended number system would be required to retain as far as possible the essential properties of the earlier number systems.

Assuming that such an extended system is possible we denote by i a solution of $x^2 + 1 = 0$. So $i^2 = -1$, and we may then solve equations like $x^2 - 2x + 2 = 0$ to find $x = 1 + i$ or $x = 1 - i$. This leads to consideration of expressions of the form $a + i\,b$ where a, b are real numbers.
If we begin again with these elements and define addition and multiplication formally we find a system which has all the algebraic properties of the earlier number system. But the system is not ordered. It is easy to explain that we are effectively operating with ordered pairs of real numbers according to rules which may be formally stated.

1. ALGEBRA

In a preliminary discussion it may be worth while to explain that in the algebra associated with number systems it is the "algebraic" properties of the numbers which is of the greatest importance. We noted that the rationals, reals, and complex numbers all had the same algebraic properties while the set of integers have most of the same properties. It may now seem useful to introduce the notion of a field in the algebraic sense and possibly also the notion of an integral domain.

(a) Polynomials

Polynomials may be formed over various sets --- and the sets usually have some kind of algebraic structure. For the various useful structures the theories of polynomials have much in common but they differ in some important points of detail. It is not feasible to ask beginners to absorb all the useful theories at the same time so we consider here only sets which have the algebraic structure of a field. More definitely the beginner may contemplate one or other of the set of rationals, the set of reals, or the set of complex numbers.

If $a_0, a_1, \ldots, a_n$ are any numbers in one of these fields the expression

$$a_0 + a_1 x + \ldots + a_n x^n$$

is called a "polynomial in x" over the field. In the most abstract approach the letter x has no meaning and the polynomial is a purely formal expression - merely a device for studying ordered finite subsets $\{a_0, \ldots, a_n\}$ of numbers. Here $a_0, \ldots, a_n$ are called the coefficients of the polynomial, x is called the "intermediate", and if $a_n \neq 0$ the polynomial is said to have degree n. We impose a structure on the set of polynomials by defining addition and multiplication formally in a way which need not be specified here. The usual algebraic laws for addition and multiplication apply. Subtraction is possible but division is not always possible. The polynomials obey the same set of algebraic laws as do the integers technically polynomials over a field form an integral domain. The zero polynomial, $a_r = 0$ all r, has no degree but apart from this the degree of a product is equal to the sum of degrees of the factors. It follows that the product of non-zero polynomials is not zero.

Ideas like divisibility and factorisation may be applied to the set of polynomials just as to the set of integers - and the theory is constructed in much the same way. Given $A = A(x)$, $B = B(x)$ $(= 0)$ any two polynomials in x, if there is a polynomial $C(x)$ such that $A = BC$ we say "B divides A" and write $B \mid A$.

We say also B is a factor of A; and of course if $C \neq 0$, C is a factor of A. The theorems

$$B \mid A \;\&\; B \mid A' \Longrightarrow B \mid (A + A')$$

and

$$B \mid A \;\&\; C \mid B \Longrightarrow C \mid A$$

follow as in arithmetic.

If $A = BC$ and if deg. $B <$ deg. A & deg. $C. <$ deg. A

then B, C are "proper" factors of A, and A is composite or reducible. If A has no proper factors we call A an irreducible polynomial or a prime polynomial. Whether or not a given polynomial is reducible depends very materially on the field over which the set of polynomials is constructed. For example it is easy to prove that $x^2 - 2$ is reducible when the underlying field is the set of real numbers:

$$x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$$

but it is irreducible when the underlying field is the set of rationals. The proof is trivial and depends only on the fact that no rational number has its square equal to 2. So $x^2 + 1$ is irreducible over the reals but is reducible over the field of complex numbers.

A polynomial of degree 1 is always prime. The polynomial $x^2 - 3x + 2 = (x-1)(x-2)$ is reducible over any field, but the polynomial.

$$x^2 + x + 1 = (x - w)(x - w^2)$$

is reducible only over fields which contain the cube roots w, w^2 of unity.

In general any polynomial of degree exceeding 1 over the complex field is reducible, and this is a statement equivalent to what used to be called the fundamental theorem of algebra. The theorem remains a fundamental one in mathematics though it is not now regarded as a theorem of algebra.

A polynomial, if not prime, can be expressed as a product of primes. The long division process of classical algebra can be used to establish the division transformation. Given two polynomials in x, say A and B suppose that deg. $A \geq$ deg. $B > 0$. We can find two polynomials q and r such that $A = qB + r$ and either $r = 0$ or deg. $r <$ deg. B. Such q, r are uniquely determined.

The H.C.F. of two polynomials A and B is a polynomial H of highest degree with highest coefficient 1 such that $H \mid A$ and $H \mid B$. Euclid's algorithm for the H.C.F. is constructed now almost exactly as in the case of the set of integers and we make similar deductions from it. Thus if A, B are given polynomials and H is their H.C.F. there are polynomials P, Q such that

$$PA + QB = H$$

If $H = 1$, we say A, B are relatively prime and

$$PA + QB = 1$$

Conversely, a relation of this kind implies that A, B are relatively prime. Again, also proved as before, if A, B are relatively prime,

$$A \mid BC \Longrightarrow A \mid C$$

Finally we may show that the factorisation of a polynomial into irreducible factors is essentially unique, i.e., apart from the introduction of "constant" factors.

(b) Polynomials as functions

Given a polynomial in the indeterminate x over a field F say

$$P(x) = a_o + a_1x + \ldots + a_n x^n$$

if we replace x by b, an element of F, we get

$$a_o + a_1b + \ldots + a_nb^n$$

This is an element of F; it is denoted by P(b)

To each element b in F the polynomial determines in this way a definite element of F, i.e., it specifies a function on F into F. Further the definitions of addition and multiplication of polynomials have been such that the polynomial relations

$$P(x) + Q(x) = R(x), \quad P(x)\,Q(x) = S(x)$$

imply the relations

$$P(b) + Q(b) = R(b), \quad P(b)\,Q(b) = S(b)$$

In consequence any relation between polynomials derived by the use of these operations yields a corresponding relation between the associated functions. Expressed briefly this means that in any relation between polynomials over F we may substitute any element of the set F for the indeterminate x. It is this principle of substitution which is so important; if it were not true any abstract theory of polynomials would be of very much less use than it is.

Consider for instance the division transformation when B = x - a, is of the first degree. Then

$$A = q(x-a) + r$$

and either r = 0, or it is a polynomial of degree 0; we may say r is an element of the field. Substituting x = b we get

$$A(b) = q(b).\ (b - a) + r,$$

a relation which holds for every b in F.

In particular, taking b = a we get A(a) = r.
So, in this division the remainder r is A(a). This is the so-called "remainder theorem". When A(a) = 0, r = 0 and we get

$$A(x) = q(x). (x-a)$$

so

$$(x - a) \mid A(x)$$

in the sense of polynomial division. Of course, the converse statement holds also. This is the "factor theorem".

Then also a is called a root of the polynomial $A(x)$. We see that a polynomial over F which is irreducible over F and of degree > 1 has no roots in F.

Again, if a polynomial $A(x)$ of degree n has n roots $a_1, \ldots, a_n$ we find the factorisation

$$A(x) = a(x - a_1) \ldots (x - a_n)$$

Hence for b in F,

$$A(b) = a(b-a_1) \ldots (b - a_n)$$

This means that for $b \neq a_i$, each i, $A(b) \neq 0$.

Hence a polynomial of degree n cannot have more than n roots. Now we may show that two different polynomials $A(x)$ and $B(x)$ cannot specify the same function in F. For then the polynomial

$$A(x) - B(x)$$

has a degree $n \geq 0$. Thus $A(b) - B(b) \neq 0$ except for at most n values of b. If F contains more than n elements then $A(b)$ and $B(b)$ are not the same functions on F.

If $A(x)$ is completely reducible to a product of n linear factors, and if it has highest coefficient 1 we have

$$A(x) = (x - a_1) \ldots (x - a_n)$$

The roots of $A(x)$ are $a_1, \ldots, a_n$; and the coefficients of $A(x)$ are

$$1, \ -\sum a_i, \ +\sum a_i a_j, \ \ldots, \ (-1)^n a_1 \ldots a_n$$

Of course if a polynomial is not completely reducible there can be no sense in speaking of a relation between its roots and its coefficients.

(c) Taylor's Theorem for Polynomials

The binomial expansion

$$(x + a)^n = x^n + nx^{n-1}a + \ldots + a^n$$

is elementary. If $f(x)$ is a polynomial

$$f = f(x) = a_0 + a_1x + \dots + a_nx^n$$

the derived polynomial $f' = Df$ is defined quite formally by

$$Df = a_1 + 2a_2 x + \dots + na_nx^{n-1}$$

Then by simple direct verification for any two polynomials f, g

$$D(f + g) = Df + Dg \ ; \ D(fg) = (Df)g + f(Dg)$$

In particular, with an obvious notation, if

$f(x) = x^n$ so

$$f' = nx^{n-1}, \ f'' = n(n-1)x^{n-2}, \ \dots \ \text{etc.,}$$

and the binomial expansion may be written

$$f(x + a) = f(x) + a\, f'(x) + \frac{a^2}{2!} f''(x) + \dots + \frac{a^n}{n!} f^{(n)}(x)$$

$$= f(a) + xf'(a) + \frac{x^2}{2!} f''(a) + \dots + \frac{x^n}{n!} f^{(n)}(a)$$

By simple addition these formulae can be extended to an arbitrary polynomial of degree n. By the "translation" $x \longrightarrow x - a$ we have

$$f(x) = f(a) + (x-a) f'(a) + \dots + \frac{(x-a)^n}{n!} f^{(n)}(a),$$

which is Taylor's theorem for polynomials (of degree n).

As we have seen if a is a root of $f(x)$, $(x-a) \mid f(x)$, $f(a) = 0$. If in addition $f'(a) = 0, \dots, f^{(r)}(a) = 0$, and $f^{(r+1)}(a) \neq 0$ we commonly say a is a root of $f(x)$ of multiplicity r. Then $(x-a)^r \mid f(x)$ and $(x-a)^{r+1} \nmid f(x)$. This follows immediately from Taylor's theorem. The converse statement is easily proved.

(d) Rational Functions. Partial Fractions

If $P(x)$, $Q(x)$ are two polynomials the expression

$$\frac{P(x)}{Q(x)}$$

is called a rational function. Of course properly speaking the word function should not be used in a purely algebraical context since the mathematical meaning of the word function is not involved at all We shall continue with the old fashioned use. The sum of two rational functions is defined by the formula

$$\frac{P_1(x)}{Q_1(x)} + \frac{P_2(x)}{Q_2(x)} = \frac{P_1Q_2 + P_2Q_1}{Q_2\ Q_2},$$

and so is a rational function. The problem of "partial fractions" is to reverse this process. We state here only the main lemma. If f is a polynomial, P, Q relatively prime polynomials and if deg. f < deg. PQ then there is a unique separation into partial fractions

$$\frac{f}{PQ} = \frac{A}{P} + \frac{B}{Q}$$

where deg. A < deg. P, deg. B < deg. Q.

Since P, Q are relatively prime we find A_1, B_1 such that

$$B_1 P + A_1 Q = 1,$$

Multiply by f and set $A = A_1 f$, $B = B_1 f$, then

$$BP + AQ = f$$

Hence

$$\frac{f}{PQ} = \frac{A}{P} + \frac{B}{Q}$$

By long division we may remove the "integral parts" of $\frac{A}{P}$ and $\frac{B}{Q}$ and then these must cancel from considerations of degree.

3. CALCULUS

(a) The Function Concept

When two variable quantities (like pressure and volume of a gas, or length and temperature of a metal bar, or distance travelled and time taken for a moving particle) are related in such a way that the value of one of them determines the value of the other we say that they are functionally related. Usually the value of either variable may be taken arbitrarily and then that of the other follows, or is determined, by the first. We speak of the dependent and independent variable. If the value of x determines that of y uniquely, we say y is a function of x and write $y = f(x)$. In mathematics we emphasise especially the formal nature of the functional relation, but it is a mistake to forget its practical origin. Formally we have simply a correspondence between values of x and values of y.

If to each element x of a set X we have associated one and only one element y of a set Y then this association or correspondence is a function defined 'on X' (its domain) and having its range in Y. If we like, the associated elements may be written as pairs (x, y): so we come to the modern definition of a function simply as a set of ordered pairs in which any particular element of X occurs in only one of the pairs. In this formulation the sets X, Y may be quite arbitrary sets and not necessarily sets of numbers. It should be recognised that this modern definition and the older forms mean the same thing.

In our calculus studies the sets X, Y are always sets of real numbers - we are concerned with real functions of real variables.

(b) Continuous Functions

The notion of continuity should be explained simply but informally. Assuming f(x) defined in an interval containing x_0 we define first what is meant by continuity at x_0. We say "f(x) is continuous at x_0 if the values which f(x) takes for values of x near x_0 are nearly equal to $f(x_0)$". A fuller discussion and a formal definition will be found in the Second Level Notes.

Continuity throughout an interval is defined as continuity at each point of the interval.

Graphs of continuous functions are continuous curves, and the general properties of continuous functions should be explained by reference to their graphs.

The fundamental properties are -

(i) if f(x) is continuous in $a \leqq x \leqq b$ there is some point x_0 in this interval such that $f(x) \leqq f(x_0)$ for all x in the interval. This statement is described by saying that a function which is continuous in a closed interval takes a greatest value in the interval. The difference between closed and open intervals should be emphasised.

(ii) if f(x) is continuous in $a \leqq x \leqq b$ and if $f(a)\,f(b) < 0$ then there is some point x_0 between a and b for which $f(x_0) = 0$.

No formal proofs of these statements would be given.

(c) Tangents to Curves

We may begin by considering continuous curves, being the graphs of continuous functions y = f(x), defined in an interval $a \leqq x \leqq b$. Considered geometrically the first problem of the calculus is that of defining what is meant by the tangent to a curve at a given point. Let P be a point on the curve at which we wish to specify the tangent. Take another point Q on the curve and consider the secant PQ. If, as Q moves towards P along the curve, the secant PQ tends to a limiting position or limiting line this line is the tangent at P to the curve. A curve will have a tangent at the point P only if this definition yields a result.

The definition is expressed in informal and geometric language and it should be regarded as a mere preliminary. The use of the description 'limiting line' almost begs the principal question. It is therefore necessary to explain just how the definition is to be understood.

Consider the slope or gradient of the secant PQ. If Q has co-ordinates x, f(x) = y, and P has co-ordinates $x_0, f(x_0)$ this slope is

$$g(x) = \frac{f(x) - f(x_0)}{x - x_0}$$

To say that the secant PQ has a limiting position must be understood to mean that the slope g(x) tends to a definite limit s, say, as $x \to x$ If this is so then the line through P with the slope s is the tangent to the curve at P. Then also we call s the slope of the curve at P.

The question then whether a given curve has a tangent at a particular point P is the same as the question whether a given function has a limit at a given point x_o — the one question is merely a geometric phrasing of the other. When the above limit s exists we say that the function f(x) is "differentiable at x_o". If f(x) is differentiable at each point of an interval then we say f(x) is differentiable throughout the interval. In this case the differential coefficient s is itself a function defined on the interval; its value at x is denoted by $f'(x)$, and this is called the derived function. In contexts where we write $f(x) = y$, we also write $f'(x) = \frac{dy}{dx}$.

With the foundations properly laid the usual discussions of the elementary text books can be quite rigorously interpreted. The usual necessary conditions and sufficient conditions for local maxima and minima may be derived. Rolle's theorem and the mean value theorem should be explained geometrically - but it may be noted here that it is quite easy to give rigorous proofs of these theorems using only the properties of continuous functions already explained. An important theoretical application of the mean value theorem is to show that if $f'(x) = 0$ in $a \leq x \leq b$ then f(x) is constant in that interval. The converse, though important, is quite trivial and would have been noted already.

(d) The Definite Integral

This is, so to speak, the 'other half' of the calculus. The theoretical problem is to define what is meant by the area of a region which is bounded wholly or partly by curves. Here it is sufficient to consider this for a region in the Cartesian plane bounded by a continous curve $y = f(x) > 0$, the x axis and ordinates at $x = a$ and $x = b$, $(a < b)$; and to suppose that f(x) is an increasing function. Divide the interval (a, b) into sub-intervals and form the sums S, s of areas of outer and inner rectangles. It is now easy to show that if all the sub-intervals have length $< \delta$ then

$$S - s < \delta \left[f(b) - f(a) \right]$$

Also that, if there is an area A satisfying our intuitive requirements then $s < A < S$.

Thus, by taking a suitably fine subdivision we see that either s or S will be a close approximation to A: and the approximation can be made as close as we wish by choice of the subdivision. Thus we may say that the area A is the limit to which either s or S tends as the subdivision is made finer and finer. This will do for the present, but it may be noted that we have not done quite what we intended. From a more sophisticated point of view we should first establish that the limits of S or s in the sense described exist and then this limit is taken as the definition of A, the area. We should then show that this definition satisfies the requirements of geometrico - physical intuition. But this more complete discussion belongs to a later stage.

Examples of the direct calculation of areas as limits in cases of the curves $y = x^2$, $y = x^3$ should be given.

The sum S (or s) may be represented symbolically in the form

$$S = \sum_a^b f(x) \Delta x$$

and the limit (which is the area) by $\int_a^b f(x)\,dx$, which is called "the definite integral of f(x) over the interval (a, b)".

The extension of the idea of a definite integral to the usual cases when f(x) is not monotonic over the whole interval or when f(x) changes sign in the interval are formal and trivial. The fundamental properties

(i) $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

(ii) if $g_1(x) \leqq f(x) \leqq g_2(x)$, $a \leqq x \leqq b$

then $\int_a^b g_1 dx \leqslant \int_a^b f\,dx \leqslant \int_a^b g_2\,dx$

are also obvious from the geometric picture. The statement (ii) contains the mean value theorem (of the integral calculus).

We now prove easily (still under the assumption that f(x) is continuous)

$$\frac{d}{dx}\int_a^x f(u)\,du = f(x)$$

which establishes the connection between differentiation and integration (and so 'joins' the two halves of the subject); this is one form of the fundamental theorem of the calculus. It leads to the calculus rule for evaluating definite integrals. The power of this calculus may now be impressed on students by showing how easily we can calculate the areas under the curves $y = x^2$ and $y = x^3$, comparing this with the direct calculations already made.

Throughout, the work should be illustrated as far as possible with simple examples and the students should work practice exercises. All the points of theoretical statements are often first fully understood only in connection with such exercises.

For example, the simplification of apparently complicated integrals by the use of relations such as

$$\int_0^a f(x)dx = \int_0^a f(a - x)\,dx$$

and of the properties of odd and even functions could be demonstrated.

Simple exercises usually serve this purpose just as well as complicated ones, but significant problems are not always simple and it is necessary to acquire useful technique. When possible there should be some intrinsically interesting and challenging problems. The following development of the theory of the elementary transcendental functions should serve to emphasise the power of the calculus methods.

(e) Exponential and Logarithmic Functions

According to the principles established the area under the curve $y = x^n$ and the ordinates at 1 and x is

$$H(x) = \int_1^x u^n du$$

which clearly holds for all $x > 1$ and for all values of n. For $n \neq -1$ we can evaluate this by the method of calculus. For $n = -1$ the method fails; this is simply because we do not know a function whose derived function is x^{-1}.

But, with $n = -1$

$$H(x) = \int_1^x \frac{du}{u}, \qquad (1)$$

and, by the general result

$$\frac{dH}{dx} = \frac{1}{x}. \qquad (2)$$

Thus the integral formula itself defines a function whose derived function is x^{-1}. We may therefore decide to study the properties of this function directly from this definition.
The first properties appear immediately:
$H(1) = 0$, $H(x)$ is an increasing function of x.

Now, using the function of a function rule,

$\frac{d}{dx} H(ax) = \frac{1}{x}$; from this $H(ax) - H(x) = \text{const.}$

Setting $x = 1$ we determine the constant and find $H(ax) = H(x) + H(a)$. Replacing x by b,

$$H(a) + H(b) = H(ab) \qquad (3)$$

This is a fundamental property of the function.

A quite similar discussion shows that

$$H(x^k) = k\ H(x) \qquad (4)$$

From (4), $H(2^x) = x\,H(2)$. This shows that $H(2^x) \to \infty$ as $x \to \infty$; and equivalently $H(x) \to \infty$ as $x \to \infty$. Again from (3), setting $a = x$, $b = \frac{1}{x}$ we find $H(x) = -H(\frac{1}{x})$, and so $H(x) \to \infty$ as $x \to 0$. The function $H(x)$ increases monotonically from $-\infty$ to $+\infty$ as x increases from 0 to ∞. $H(x) \gtrless 0$ as $x \gtrless 1$, and the gradient of H(x) continually decreases as x increases. The graph of H(x) may now be drawn showing these features.

Since H(x) is continuous and increasing there is a unique value e such that $H(e) = 1$. If we write for a moment $x = e^y$ then by (4)

$$H(x) = H(e^y) = y\ H(e) = y = \log_e x$$

by the usual definition of a logarithm. This logarithm, to base e, is called the natural logarithm of x. From this we have its essential properties.

$$\log_e x = \int_1^x \frac{du}{u}, \quad \frac{d}{dx}(\log_e x) = \frac{1}{x}.$$

Further, since $\sqrt{x} < x$, for $x > 1$, we have

$$\log_e x = \int_1^x \frac{du}{u} < \int_1^x \frac{du}{\sqrt{u}} < 2\sqrt{x},$$

and so

$$\frac{\log x}{x} \longrightarrow 0 \quad \text{as} \quad x \longrightarrow \infty.$$

The relations $y = \log_e x$, $x = e^y$ are equivalent and so

$$\frac{d}{dy}(e^y) = \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = x = e^y$$

or, what is the same thing, $\frac{d}{dx}(e^x) = e^x$.

Finally we may obtain the series expansions of e^x and $\log(1 + x)$. A simple derivation of the exponential series can be given by using the following principle. Suppose $v_o = v_o(x)$ be (say) a positive function defined over some range (O, X); and that $o < v_o(x) < K$.

If we define a sequence $v_1(x)$, $v_2(x)$, ... in (O, X) by

$$V_{n+1} = \int_o^x v_n \, dx$$

then we find, by induction, $0 < V_n < \frac{Kx^n}{n!} < \frac{KX^n}{n!}$

Thus $v_n \to 0$ as $n \to \infty$. Now take $v_o = e^x$.

Then $v_1 = e^x - 1 = v_o - 1$, or $v_o - v_1 = 1$.

Then by integration over (0, x) we get successively

$$v_1 - v_2 = x$$

$$v_2 - v_3 = \frac{x^2}{2!}$$

$$v_n - v_{n+1} = \frac{x^n}{n!}$$

By addition and transposition of terms

$$e^x - \left[1 + x + \dots + \frac{x^n}{n!}\right] = v_{n+1} \to 0$$

as $n \longrightarrow \infty$. This establishes the exponential series

$$e^x = 1 + x + \dots + \frac{x^n}{n!} + \dots$$

We have supposed $x > 0$; but the proof is easily extended to the case $x < 0$. By setting $x = 1$ we easily compute $e = 2.718282 \ldots$

From the G.P. formula

$$\frac{1}{1-x} = 1 + x + \ldots + x^{n-1} + \frac{x^n}{1-x}$$

we find by integration and then letting $n \longrightarrow \infty$

$$-\log(1-x) = x + \frac{x^2}{2} + \ldots + \frac{x^n}{n} \quad \ldots$$

valid in $-1 \leq x < 1$.

(f) Arcs of Curves

Like areas, the length of arc of a curved line is a matter of definition. Given a curve $y = f(x)$ consider the arc AB between $x = a$ and $x = b$

Select points $A = P_o, P_1, \ldots, P_n = B$ in order on the curve and form the sum of the lengths of the chords $P_{i-1} P_i$,

$$\sum_{i=1}^{m} P_{i-1} P_i$$

The limit of this sum as the division is made finer and finer, if it exists, is by definition the length of the arc AB. Under certain conditions, viz. $f(x)$ has a continuous derivative, it is easy to show that the limit does exist and is represented by the definite integral.

$$\int_a^b \left[1 + \left\{ f'(x) \right\}^2\right]^{\frac{1}{2}} dx = \int_a^b \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx.$$

Thus if P, P' be any adjacent points in the above sum with co-ordinate differences Δx, Δy then

$$PQ^2 = (\Delta x)^2 + (\Delta y)^2, \text{ and } \frac{\Delta y}{\Delta x} = f'(x)$$

where x is a number between the abscissae of P, Q. This uses the mean values theorem. So setting

$$\varphi = \sqrt{1 + (f')^2}$$

we find

$$PQ = \varphi(x). \quad \Delta x.$$

Then for the above sum we have

$$\sum P_{i-1} P_i = \sum \varphi(x) \quad \Delta x$$

As the division is made finer and finer the sum on the right has the limit $\int_a^b \varphi(x)\,dx$

by the definition of the integral. This is the result.

For the unit circle $x^2 + y^2 = 1$, we find

$$x\frac{dx}{dy} + y = 0,$$

$$1 + \left(\frac{dx}{dy}\right)^2 = \frac{1}{x^2}$$

Hence the length of arc θ between (1,0) and (x, y) in the first quadrant is

$$\theta = \int_0^y \frac{dy}{x}$$

Now, by definition θ is the angle subtended by this arc at the origin. Also, by definition, the trigonometric functions $\cos\theta$ and $\sin\theta$ are

$$\cos\theta = x, \ \sin\theta = y.$$

Then $$\frac{dy}{d\theta} = \frac{1}{\frac{d\theta}{dy}} = \frac{1}{\frac{1}{x}} = x,$$

or $\frac{d}{d\theta}(\sin\theta) = \cos\theta$. So we have the definitions and the differentiation of the trigonometric functions all in a few lines. From these all the other properties are easily derived.

The series expansions for sin x and cos x may be found by the method used for e^x. We set

$$u_o = \cos x, \qquad U_n+1 = \int_o^x U_n \ dx.$$

Then as before $u_n \longrightarrow 0$ as $n \longrightarrow \infty$ Now we find easily

$$u_o + u_2 = 1, \qquad u_1 + u_3 = x$$

$$u_2 + u_4 = \frac{x^2}{2!}, \qquad u_3 + u_5 = \frac{x^3}{3!}$$

$$\vdots \qquad\qquad \vdots$$

etc. Alternating the signs, adding the first n rows, and then letting $n \to \infty$ we have rigorously established the expansions for sin x and cos x.

The elementary theory is completed with the definition of the inverse circular functions, $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$; with the selection of principal values and with their differentiation. Further details need not be given here. Perhaps enough has been written to

indicate the advantages of first establishing completely the fundamental principles of the calculus and using these for the development of the theory of the elementary functions from appropriate definitions.

(g) Integration

We are now in a position to set up a table of standard elementary indefinite integrals. Introduction of the methods of change of variable and integration by parts facilitates the use of the calculus in application. Naturally in a first course only simple examples would be worked; it is far more important and of greater interest to teach the essential principles of methods and applications.

. PLANE GEOMETRY, GEOMETRIC ALGEBRA, MATRICES, GEOMETRY IN THREE DIMENSIONS

(a) Elementary Plane Analytical Geometry

(As in Second Level (2F), items 4 and 7 (e), (f). The theme at the end of item 7 (f), should be developed further.)

Determinants etc.

This subject can be treated first here and in more detail at the end of item 4 (b).

The two lines

$$a_1x + b_1y + c_1 = 0$$

$$a_2x + b_2y + c_2 = 0$$

(assumed to be not parallel) meet in the point

$$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \quad \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$$

The value of the linear expression

$$a_3x + b_3y + c_3$$

at this point is Δ/C,

where $\Delta = a_3(b_1c_2 - b_2c_1) + b_3(c_1a_2 - c_2a_1)$
$+ c_3(a_1b_2 - a_2b_1)$,

and $C = a_1b_2 - a_2b_1$.

Δ is the sum of the six terms $\pm a_i b_j c_k$

where ijk is an arrangement of 123 and the sign is negative for the three terms in which ijk is obtained from 123 by a single interchange of two digits, and positive for the others.

$\triangle$ is written as
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix};$$

with the corresponding conventions, C can be written as

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

The two possible signs of the value of $\triangle$ correspond to the two regions into which the line $a_3x + b_3y + c = 0$ divides the plane (i.e., the intersection of the given two lines lies to one or other side of this line according as $\triangle/C$ is positive or negative). $\triangle = 0$ is a necessary and sufficient condition that the three lines should be concurrent.

A necessary and sufficient condition that three equations

$$a_1p + b_1q + c_1r = 0$$

$$a_2p + b_2q + c_2r = 0$$

$$a_3p + b_3q + c_3r = 0$$

should have a solution (other than $p = q = r = 0$) (in the ratios $p : q : r$) is that $\triangle = 0$

From the section formulae, if $(x_1 y_1)$, (x_2y_2), (x_3, y_3) are collinear, there exist multipliers l, m, such that

$$lx_1 + mx_2 = (l + m)\, x_3$$

$$ly_1 + my_2 = (l + m)\, y_3$$

These equations can be written

$$x_1l + x_2m + x_3n = 0$$

$$y_1l + y_2m + y_3n = 0$$

$$l + m + n = 0$$

and therefore a necessary and sufficient condition for the three points to be collinear is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Compare the condition obtained from assuming that the three points lie on the line $ax + by + c = 0$.

The area of the triangle

$\{(x_1, y_1), (x_2, y_2), 0\}$ is $\frac{1}{2}(x_1y_2 - x_2y_1)$

The sense of description of the triangle and its relation to the sign of this expression should be explained.

Area of $\{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$

$$= \text{area of } \{(x_1 - x_3, y_1 - y_3), (x_2 - x_3, y_2 - y_3), (0,0)\}$$
$$= \tfrac{1}{2}\left[(x_1 - x_3)(y_2 - y_3) - (x_2 - x_3)(y_1 - y_3)\right]$$
$$= \tfrac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}$$

(b) Linear Transformations in the Plane,

Introduction of Matrix Algebra

These notes form a first draft of a course aimed at teaching students about matrices in a way which presents matrix algebra as a living piece of mathematics which grew out of attempts to provide solutions to mathematical problems in a form in which the true nature of the problem was not overwhelmed by a mass of complicated and repetitive algebraic manipulation.

For a first course, where facility in matrix manipulation has to be acquired, as well as understanding of the matrix operations, it seems best to restrict the geometrical problems on which the matrices are used to the Euclidean vector plane and the Euclidean three-dimensional vector space (in each of which an origin is kept fixed). The plane alone would afford plenty of scope, but the extension to three dimensions is well worth the effort because of the need there is to give as much practice as possible in thinking in terms of three dimensional images.

Much fuller notes, together with exercises, are expected to be available before the end of 1965.

A. Transformations of the plane

In relation to a fixed origin O we define a set of geometrical operations each of which associates with each point P in the plane another point P', i.e., we transform the plane into itself or map the plane into itself.

(i) Rotations ("Turns")

For a given origin O and an angle of given magnitude a, the point P' associated with each point P is defined by:

$$\angle^* POP' = \alpha$$

$$^* OP' = {}^*OP$$

Write this relation between P and P' as

$$P' = \mathcal{T}_\alpha \quad P$$

(Initially we may write write $\mathcal{T}_\alpha$ (P), since $\mathcal{T}_\alpha$ is the symbol of a geometric function of the points P , but the parentheses can be omitted without introducing ambiguity — partly because of the different founts used for $\mathcal{T}$ and P).

$\mathcal{T}_\alpha$ is the symbol representing the operation of rotation which transforms P into P'. We refer to $\mathcal{T}_\alpha$ as an operator. From geometry we have

$$\mathcal{T}_\beta \quad (\mathcal{T}_\alpha P) = \mathcal{T}_{\alpha+\beta} = \mathcal{T}_\alpha \quad (\mathcal{T}_\beta P)$$

Note the order of operations and operators :

"$\mathcal{T}_\alpha$" followed by "$\mathcal{T}_\beta$" has to be written $\mathcal{T}_\beta$ ($\mathcal{T}_\alpha$ P), and again we may omit the parentheses and write $\mathcal{T}_\beta \mathcal{T}_\alpha$ P. In this case of course the operator $\mathcal{T}_\beta \mathcal{T}_\alpha$, which is "the resultant of $\mathcal{T}_\alpha$ followed by $\mathcal{T}_\beta$" (or "the result of compounding $\mathcal{T}_\beta$ with $\mathcal{T}_\alpha$ ") is the same as $\mathcal{T}_\alpha \mathcal{T}_\beta$.

Since the relation

$$\mathcal{T}_\alpha \mathcal{T}_\beta \quad P = \mathcal{T}_{\alpha + \beta} \, P$$

is valid for all points P in the plane, we could write it simply as a relation among the operators.

$$\mathcal{T}_\beta \mathcal{T}_\alpha = \mathcal{T}_{\alpha+\beta} = \mathcal{T}_\alpha \mathcal{T}_\beta$$

Now introduce the identity operator $\mathcal{I}$, with the property

$$\mathcal{I} P = P$$

for all P. Then from geometry

$$\mathcal{T}_{-\alpha} \mathcal{T}_\alpha = \mathcal{I}$$

and we can introduce the inverse operator $\mathcal{T}_\alpha^{-1}$ with the definition

$$\mathcal{T}_\alpha^{-1} = \mathcal{T}_{-\alpha}$$

$\mathcal{T}_\alpha^{-1}$ is the unique rotation which compounded with $\mathcal{T}_\alpha$ produces $\mathcal{I}$.

(Could follow now with the set of rules for combining rotations, including

$$\mathcal{T}_{\alpha+2\pi} = \mathcal{T}_\alpha, \quad (\mathcal{T}_\alpha)^n = \mathcal{T}_{n\alpha}$$

See also subsection B, p.31)

(ii) <u>Reflections ("Symmetries")</u>

In relation to a given line a through O we define the operation $\mathcal{S}_a$ of reflection in such a way that, for any point P,

$$P' = \mathcal{S}_a P$$

when PP' $\perp$ a

and a bisects PP'

P' is the reflection of P in a.

Properties.

1. $\mathcal{S}_a(\mathcal{S}_a P) = P$ for all P,

i.e., $\mathcal{S}_a^{\,2} = \mathcal{I}$

or $\mathcal{S}_a^{-1} = \mathcal{S}_a$

2. $\mathcal{S}_b \mathcal{S}_a = \mathcal{T}_{2\angle^* ab}$

where $\angle^* ab$ is the measure of the angle from a to b, and $0 \leq \angle ab < \pi$. $\angle^* ab + \angle^* ba = \pi$

$$\mathcal{S}_a \mathcal{S}_b = \mathcal{T}_{2\angle^* ba} = \mathcal{T}_{2(\pi - \angle^* ab)} = \mathcal{T}_{-2\angle^* ab} = \left(\mathcal{T}_{2\angle^* ab}\right)^{-1} = \left(\mathcal{S}_b \mathcal{S}_a\right)^{-1}$$

In fact, since

$\mathcal{S}_a^{\,2} = \mathcal{I}$, $\mathcal{S}_a \mathcal{S}_b \mathcal{S}_b \mathcal{S}_a = \mathcal{I}$ and therefore

$$\mathcal{S}_a \mathcal{S}_b = (\mathcal{S}_b \mathcal{S}_a)^{-1}$$

3. Given a, b, c through O there is a single line d through O such that

$$\mathcal{S}_c \mathcal{S}_b \mathcal{S}_a = \mathcal{S}_d$$

(d is defined by $\angle^* ab = \angle^* dc$)

$$\mathcal{S}_a \mathcal{S}_b \mathcal{S}_c = \mathcal{S}_c \mathcal{S}_b \mathcal{S}_a$$

4. The fixed points under the operation $\mathcal{S}_a$ are the points of the line a, i.e.,

$$\mathcal{S}_a K = K \Longleftrightarrow K \in a .$$

The line a is <u>point-by-point</u> (<u>pointwise</u>) <u>invariant</u>.

If $h \perp a$, and $H \in h$, then

$$H' = \mathcal{S}_a H \Longrightarrow H' \in h$$

and conversely, so that h, $h \neq a$, is overall invariant under $\mathcal{S}_a$ if and only if $h \perp a$.

(iii) <u>Translations ("Displacements")</u>

If A B are any two points they determine an interval $\ulcorner AB \urcorner$ and a transformation $\mathcal{D}_{AB}$ of the plane in which, for any point P,

$$P' = \mathscr{D}_{AB} P$$

is defined by $PP' \,//\, AB$ & $PA \,//\, P'B$

Properties 1. $\mathscr{D}_{BC} \; \mathscr{D}_{AB} = \mathscr{D}_{AC} = \mathscr{D}_{AB} \; \mathscr{D}_{BC}$

2. $\mathscr{D}_{AB} \; \mathscr{D}_{BA} = \mathscr{I}$

$\mathscr{D}^{-1}_{AB} = \mathscr{D}_{BA}$

3. $\mathscr{D}_{AB} \; \mathscr{S}_a = \mathscr{S}_a \; \mathscr{D}_{A'B'}$

where $A' = \mathscr{S}_a A$, $B' = \mathscr{S}_a B$.

(iv) Congruence Transformations

The effect of an operator $\mathscr{T}$, $\mathscr{S}$ or $\mathscr{D}$ on any geometrical figure is to produce another geometrical figure, which, from geometrical considerations, is congruent to the original figure. The operators $\mathscr{T}$, $\mathscr{S}$ and $\mathscr{D}$ can therefore be described as generating transformations of the plane into itself or maps of the plane into itself, and in particular to generate congruence transformations.

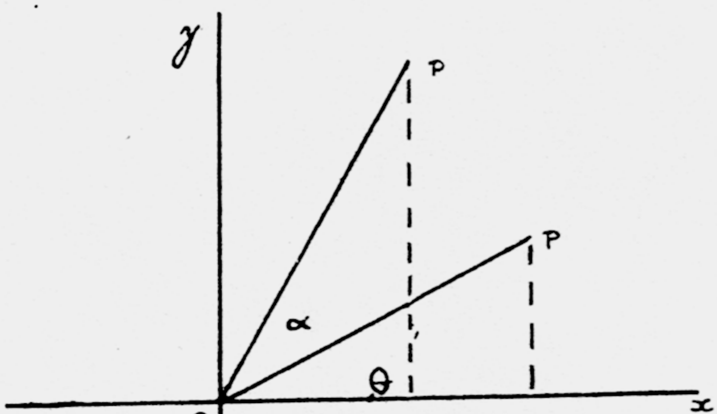
The explicit properties which remain invariant under these transformations are:

a point	transforms into	a point
a line		a line
parallel lines		parallel lines
perpendicular lines		perpendicular lines
an interval of length d		an interval of length d
an angle of magnitude α		an angle of magnitude α but sense is reversed by reflection

B. Cartesian form of rotations, reflections,

2 x 2 Matrices

(i) Rotation:



$$P \quad \text{is} \quad (r \cos\theta, \; r \sin\theta)$$

$$P' \quad \text{is} \quad (r\cos(\theta+\alpha), \; r\sin(\theta+\alpha))$$

so that $x' = r(\cos\theta \cos\alpha - \sin\theta \sin\alpha)$

$y' = r(\cos\theta \sin\alpha - \sin\theta \cos\alpha)$

i.e., $x' = \cos\alpha . x - \sin\alpha . y$

$$y' = \sin\alpha . x + \cos\alpha . y \qquad (1)$$

Let us invent an algebraic operator T_α, so that we could write these equations in the form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad (2)$$

or $\quad \underline{\underline{r}}' = \underline{\underline{T}}_\alpha \underline{\underline{r}}$,

where $\underline{\underline{r}} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\underline{\underline{r}}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$

are <u>vectors</u>, <u>column-vectors</u>, or <u>column-matrices</u> and

$$\underline{\underline{T}}_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

is the <u>matrix</u> of two <u>rows</u> $\begin{bmatrix} \cos\alpha , & -\sin\alpha \end{bmatrix}$,
$\begin{bmatrix} \sin\alpha , & \cos\alpha \end{bmatrix}$. and two <u>columns</u>

$$\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}, \qquad \begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix}$$

The symbolic relation (3) is a condensed form of (2) which in turn corresponds exactly to the two explicit equations (1).

The matrix $\underline{\underline{T}}_\alpha$ is an <u>algebraic operator</u> transforming

the vector $\underline{\underline{r}}$ into the vector $\underline{\underline{r}}'$

Let us follow the next step in the account of the geometric operators to discover how to combine the algebraic operators. I.e., consider the algebraic equivalent of

$$P'' = \mathcal{T}_\beta P' = \mathcal{T}_\beta \mathcal{T}_\alpha P.$$

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= \begin{bmatrix} \cos_\beta & -\sin_\beta \\ \sin_\beta & \cos_\beta \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} \cos_\beta \cos\alpha - \sin_\beta \sin\alpha & \cos\beta\,(-\sin)\alpha) - \sin_\beta \cos_\alpha \\ \sin_\beta \cos\alpha + \cos_\beta \sin\alpha & \sin\beta\,(-\sin\alpha) + \cos_\beta \cos_\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta & \cos(\alpha+\beta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

This is exactly $T_\beta . T_\alpha r = T_{\alpha+\beta} r$ and gives the rules which have to be followed if the "matrices" T_α are to correspond exactly to the geometric operators $\mathcal{T}_\alpha$. This is the line of thought which led Cayley to the invention of matrices to represent the operators and matrix multiplication, to represent the compounding of two operators.

$$\underline{\underline{T}}_o = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underline{\underline{1}} \text{, the unit matrix.}$$

$$\underline{\underline{T}}_{\frac{1}{2}\pi} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \underline{\underline{T}}_{\pi} = -\underline{\underline{1}} .$$

The rule for multiplication shows up most clearly if we use double subscripts. Take in a matrix $\underline{\underline{A}}$, a_{ij} to be the element in the row numbered i and column numbered j, so that $\underline{\underline{A}}$ can be written as

$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Then

$$\begin{array}{ccc} & \text{column } 1 & \text{column } 2 & \\ \underline{\underline{A}}\,\underline{\underline{B}} = & \left[\begin{array}{c} a_{11}b_{11} + a_{12}b_{21} \\ a_{21}b_{11} + a_{22}b_{21} \end{array}\right. & \left.\begin{array}{c} a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{22} + a_{22}b_{22} \end{array}\right] & \begin{array}{l} \text{row } 1 \\ \text{row } 2 \end{array} \end{array}$$

or, if $\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{C}}$, then

$$c_{rs} = a_{r1}\, b_{1s} + a_{r2}\, b_{2s}$$

clearly in general $\underline{\underline{A}}\,\underline{\underline{B}} \neq \underline{\underline{B}}\,\underline{\underline{A}}$, although

$$\underline{\underline{T}}_\beta \underline{\underline{T}}_\alpha = \underline{\underline{T}}_\alpha \underline{\underline{T}}_\beta \text{ (but } \underline{\underline{S}}_b\, \underline{\underline{S}}_a \neq \underline{\underline{S}}_a\, \underline{\underline{S}}_b\text{.)}$$

Matrix multiplication is associative.

(ii) Reflections

If $y = x \tan \theta$ is the line l, write $\mathcal{S}_\theta$

for $\mathscr{S}_1$, and $\underline{\underline{S}}_\theta$ for the matrix corresponding to $\mathscr{S}_\theta$. Then

$$\underline{\underline{S}}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \underline{\underline{S}}_{\frac{1}{4}\pi} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad S_{\frac{1}{2}\pi} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\underline{\underline{S}}_\theta$ can be computed directly from the Cartesian diagram, but it is simpler to use the relation

$$\mathscr{S}_\theta \, \mathscr{S}_0 = \mathscr{T}_{2\theta},$$
$$\mathscr{S}_\theta = \mathscr{T}_{2\theta} \, \mathscr{S}_0,$$

so that

$$\underline{\underline{S}}_\theta = \underline{\underline{T}}_{2\theta} \, \underline{\underline{S}}_0 = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

If we use the perpendicular form of the equation of the line:

$$x \cos\varphi + y \sin\varphi = 0, \text{ where } 0 = \tfrac{1}{2}\pi + \varphi,$$

we have

$$\underline{\underline{S}}_{\frac{1}{2}\pi + \varphi} = \begin{bmatrix} -\cos 2\varphi & -\sin 2\varphi \\ -\sin 2\varphi & \cos 2\varphi \end{bmatrix}$$

(iii) <u>The Affine Transformation</u>

This transformation we define algebraically as a generalization of $\underline{\underline{S}}$ and $\underline{\underline{T}}$, and then investigate its geometric properties. The discussion depends on the section formulae

$$x_3 = \frac{k_1 x_1 + k_2 x_2}{k_1 + k_2}, \quad y_3 = \frac{k_1 y_1 + k_2 y_2}{k_1 + k_2}$$

which we can write as

$$\begin{bmatrix} (k_1 + k_2)\, x_3 \\ (k_1 + k_2)\, y_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

or $\quad (k_1 + k_2)\, \underline{\underline{r}}_3 = \begin{bmatrix} \underline{\underline{r}}_1 & \underline{\underline{r}}_2 \end{bmatrix} \underline{\underline{k}}$

The transformation is

$$\underline{\underline{r}}' = \underline{\underline{M}}\, \underline{\underline{r}}$$

where $\quad \underline{\underline{M}} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$

When this is applied to the points of a line

$$(k_1 + k_2) \;\underline{\underline{r}} = \begin{bmatrix} r_1 & r_2 \end{bmatrix} k,$$

(k_1, k_2 vary along the line) we find

$$\underline{\underline{M}} \left\{(k_1 + k_2)\; \underline{\underline{r}}\right\} = \underline{\underline{M}} \begin{bmatrix} \underline{\underline{r}}_1 & \underline{\underline{r}}_2 \end{bmatrix} \underline{\underline{k}},$$

which we can rewrite as

$$(k_1 + k_2)\; \underline{\underline{M}}\; \underline{\underline{r}} = \begin{bmatrix} \underline{\underline{M}}\, \underline{\underline{r}}_1 & \underline{\underline{M}}\, \underline{\underline{r}}_2 \end{bmatrix} k$$

i.e., the transformed points are the points of

the line $\underline{\underline{M}}\, \underline{\underline{r}}_1$, $\underline{\underline{M}}\, \underline{\underline{r}}_2$

Write $\mathcal{M}$ for the corresponding geometric transformation. Then, if $\mathcal{M}$ $P_2 = P_2'$ and $P_3 \in P_1 P_2$, we have

$P'_3 \in P_1' \; P_2'$ and

$$*P_1 P_3 / *P_1 P_2 = *\, P_1' P'_3 \;/\; *P_1' P'_2$$

Thus, under the transformation $\mathcal{M}$,

points	become	points
lines	become	lines

ratios of displacements on a line become equal ratios on the transformed line so that

parallels	become	parallels

But

distances are altered and

angles are altered

C. <u>Some properties of Matrices under Multiplication</u>

(i) <u>The inverse matrix</u>

$\underline{\underline{M}}^{-1}$ is defined by $\underline{\underline{M}}^{-1}\; \underline{\underline{M}} = \underline{\underline{1}}$

If $\underline{\underline{M}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\underline{\underline{M}}^{-1} = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$

we have

$$\begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left.\begin{array}{l} ax + cy = 1 \\ bx + dy = 0 \end{array}\right\} \qquad \left.\begin{array}{l} az + ct = 0 \\ bz + dt = 1 \end{array}\right\},$$

So that
$$\underline{\underline{M}}^{-1} = \begin{bmatrix} \dfrac{d}{ad - bc} & \dfrac{-b}{ad - bc} \\ \dfrac{-c}{ad - bc} & \dfrac{a}{ad - bc} \end{bmatrix}$$

$$\underline{\underline{M}}\,\underline{\underline{M}}^{-1} = \underline{\underline{M}}^{-1}\underline{\underline{M}} = \underline{\underline{1}}$$

Thus $\underline{\underline{M}}^{-1}$ exists if and only if $ad - bc \neq 0$

If $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are any two non-singular matrices

$$(\underline{\underline{A}}\,\underline{\underline{B}})^{-1} = \underline{\underline{B}}^{-1}\,\underline{\underline{A}}^{-1}.$$

(ii) <u>The determinant of a matrix</u>

Notation: $ad - bc = \det \underline{\underline{M}} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

If $\det \underline{\underline{M}} = 0$, the matrix is <u>singular</u> (i.e. it has no inverse).

If the matrix $\underline{\underline{M}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is singular,

the affine transformation

$$\underline{\underline{M}} \begin{bmatrix} x_o \\ y_o \end{bmatrix} = \begin{bmatrix} ax_o + by_o \\ cx_o + dy_o \end{bmatrix} = \begin{bmatrix} ax_o + by_o \\ (ax_o + by_o)\,c/a \end{bmatrix}$$

(provided $a \neq 0$), is such that for all points r the transformed $\underline{\underline{M}}\,\underline{r}$ lies on the line $cx - ay = 0$, i.e., the <u>transformation is singular</u>, the whole plane being <u>mapped onto</u> the line.

Note that $\det \underline{\underline{T}} = 1$, $\det \underline{\underline{S}} = -1$, so that these matrices are always non-singular.

By direct multiplication we find for any two matrices $\underline{\underline{M}}$, $\underline{\underline{M}}'$

$$\det(\underline{\underline{M}}\,\underline{\underline{M}}') = \det \underline{\underline{M}} \det \underline{\underline{M}}' = \det(\underline{\underline{M}}'\,\underline{\underline{M}})$$

$$\det(\underline{\underline{M}}^{-1}) = (\det \underline{\underline{M}})^{-1}$$

(iii) <u>The zero vector and zero matrix</u>

$$\underline{\underline{\dot{o}}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \underline{\underline{0}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

For all $\underline{\underline{M}}$, $\underline{\underline{M}}\,\underline{\underline{o}} = \underline{\underline{o}}$ and $\underline{\underline{M}}\,\underline{\underline{0}} = \underline{\underline{0}}\,\underline{\underline{M}} = \underline{\underline{0}}$

The most general singular matrix may be written as

$$\begin{bmatrix} rr' & rs' \\ r's & ss' \end{bmatrix}$$

For this matrix we have

$$\begin{bmatrix} rr' & rs' \\ r's & ss' \end{bmatrix} \begin{bmatrix} us' \\ -ur' \end{bmatrix} = \underline{\underline{o}} \quad \text{for any } u$$

and
$$\begin{bmatrix} rr' & rs' \\ r's & ss' \end{bmatrix} \begin{bmatrix} us & vs' \\ -ur' & -vr' \end{bmatrix} = \underline{\underline{0}} \quad \text{for any } u,\ v$$

and
$$\begin{bmatrix} hs & -hr \\ ks & -kr \end{bmatrix} \begin{bmatrix} rr' & rs' \\ r's & ss' \end{bmatrix} = \underline{\underline{0}} \quad \text{for any } h,\ k$$

Thus in matrix algebra there are divisors of the zero matrix, i.e., there exist pairs of matrices $\underline{\underline{M}}$ and $\underline{\underline{N}}$, both non-zero such that $\underline{\underline{M}}\,\underline{\underline{N}} = \underline{\underline{0}}$. If we are given $\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{0}}$ we cannot deduce that either $\underline{\underline{A}}$ or $\underline{\underline{B}}$ is the zero matrix, but only that, if neither $\underline{\underline{A}}$ nor $\underline{\underline{B}}$ is the zero matrix, both are singular,

(iv) The transposed vector and matrix

If $\underline{\underline{a}} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ we write $\underline{\underline{a}}^T = \begin{bmatrix} a_1, & a_2 \end{bmatrix}$

for the transpose of $\underline{\underline{a}}$, i.e., the row-vector with components identical with those of the column vector $\underline{\underline{a}}$. Likewise

if $\underline{\underline{M}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \underline{\underline{M}}^T \overset{\text{def}}{=} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

If $\underline{\underline{A}}$ is any non-singular matrix:

$$(\underline{\underline{A}}^T)^{-1} = (\underline{\underline{A}}^{-1})^T$$

If $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are any two matrices, and $\underline{\underline{k}}$ any vector

$$(\underline{\underline{A}}\,\underline{\underline{B}})^T = \underline{\underline{B}}^T\ \underline{\underline{A}}^T$$
$$(\underline{\underline{A}}\,\underline{\underline{k}})^T = \underline{\underline{k}}^T\ \underline{\underline{A}}^T$$

In particular $\underline{\underline{S}}^T = \underline{\underline{S}}^{-1} = \underline{\underline{S}}$, $\quad \underline{\underline{T}}^T = \underline{\underline{T}}^{-1}$

(v) Uniqueness of $\underline{\underline{S}}$ and $\underline{\underline{T}}$

Theorem

If under an affine transformation, with fixed origin, distance is invariant, then the transformation is either a rotation or a reflection

We have to have

$$x'^2 + y'^2 \equiv x^2 + y^2$$

i.e. $\underline{r}'^{T} \; \underline{r}' = \underline{r}^{T}\underline{r}$

i.e. $(\underline{\underline{M}}\,\underline{r})^{T}(\underline{\underline{M}}\,\underline{r}) = \underline{r}^{T}\underline{r}$

i.e. $\underline{r}^{T}(\underline{\underline{M}}^{T}\underline{\underline{M}})\,\underline{r} = \underline{r}^{T}\underline{1}\,\underline{r}$

$$\underline{\underline{M}}^{T}\underline{\underline{M}} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{bmatrix}$$

$$\underline{r}^{T}\,\underline{\underline{M}}^{T}\underline{\underline{M}}\,\underline{r} = (a^2+c^2)x^2 + 2\,(ab+cd)xy + (b^2+d^2)y^2$$

$$\underline{r}^{T}\,\underline{r} = x^2 + y^2$$

Thus, if distance is invariant,

$$\left.\begin{aligned} a^2 + c^2 &= 1 \\ b^2 + d^2 &= 1 \end{aligned}\right\} \qquad ab + cd = 0$$

Take $a = \cos\alpha$, $c = \sin\alpha$,

$b = \cos\beta$, $d = \sin\beta$,

then $\cos(\alpha - \beta) = 0$

i.e., either $\beta = \alpha - \frac{1}{2}\pi$ or $\beta = \alpha + \frac{1}{2}\pi$

and the two possible matrices are

$$\begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix} = \underline{\underline{S}}_{\frac{1}{2}\alpha}$$

or
$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \underline{\underline{T}}_{\alpha}$$

A matrix with the property $\underline{\underline{M}}^{T}\underline{\underline{M}} = \underline{1}$, i.e.,
$\underline{\underline{M}}^{T} = \underline{\underline{M}}^{-1}$, is called orthogonal.

D. Displacement, Matrix Addition, and Matrix Algebra

If H is (h, k) and $\mathcal{D}_{OH}P = P'$

where P is (x, y) and P' is (x', y'), we have

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + h \\ y + k \end{bmatrix}$$

which we write as
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} h \\ k \end{bmatrix}$$

or as
$$\underline{r}' = \underline{r} + \underline{h}$$

This suggests an addition operation for vectors, and then immediately an addition operation for matrices.

If $\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\underline{\underline{B}} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

We define $\underline{\underline{A}} + \underline{\underline{B}}$ by:

$$\underline{\underline{A}} + \underline{\underline{B}} \overset{def}{=} \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

Thus $\underline{\underline{A}} + \underline{\underline{A}} = \begin{bmatrix} 2a_{11} & 2a_{12} \\ 2a_{21} & 2a_{22} \end{bmatrix} \overset{def}{=} 2A;$

define similarly $n\underline{\underline{A}}$, then if $\underline{\underline{B}} = n\underline{\underline{A}}$, $\underline{\underline{A}} = \frac{1}{n}\ \underline{\underline{B}}$

and so through rational to real <u>scalar multiples</u> of $\underline{\underline{A}}$.

For any number k, $k\underline{\underline{A}} = \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$\det (k\underline{\underline{A}}) = k^2 \det \underline{\underline{A}}.$$

Definition of $\underline{\underline{A}}-\underline{\underline{B}}$, and $\underline{\underline{A}}-\underline{\underline{A}} = \underline{\underline{0}}$ (zero matrix).

Combination of sums and products.

The rules of matrix algebra are the same as those for a field except that:

1. Multiplication is not commutative
2. there are divisors of zero, i.e. not every element has a multiplicative inverse.

E. <u>The characteristic function, eigenvalues, eigenvectors, the Cayley-Hamilton theorem</u>

(i) The Characteristic function of $\underline{\underline{A}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$\det (x\underline{\underline{1}} - \underline{\underline{A}}) = x^2 - (a + d)\, x + \det \underline{\underline{A}}.$$

The roots of $\det (x\underline{\underline{1}} - \underline{\underline{A}}) = 0$ are the <u>eigenvalues</u>; for these values the vectors such that

$(x\underline{\underline{1}} - \underline{\underline{A}})\, \underline{u} = \underline{o}$ are the <u>eigenvectors</u>

(ii) <u>C-H Theorem:</u>

The matrix $\underline{\underline{A}}$ satisfies $\underline{\underline{A}}^2 - (a + d)\, \underline{\underline{A}} + (\det A)\, \underline{\underline{1}} = \underline{\underline{0}}$

Thus in forming matrix polynomials

$\underline{\underline{A}}^2$ may be replaced by $(a + d)\underline{\underline{A}} - (\det \underline{\underline{A}})\, \underline{\underline{1}}$

$$\underline{\underline{A}}^3 \text{ by } (a + d) \left\{ (a + d)\, \underline{\underline{A}} - \det \underline{\underline{A}})\underline{\underline{1}} \right\} - (\det \underline{\underline{A}})\underline{\underline{A}} \text{ etc.}$$

In particular, for $\underline{\underline{M}} = \begin{bmatrix} rr' & rs' \\ r's & ss' \end{bmatrix}$,

$$\underline{\underline{M}}^2 = (rr' + ss')\quad \underline{\underline{M}}.$$

Discuss these also in relation to the geometric series.
(Cf end of item VI.)

F. Changes of co-ordinates

(i) Parallel shift of axes.

$$\underline{\underline{r}}' = \underline{\underline{r}} + \underline{\underline{h}}$$

may be interpreted either as a transformation of the plane in which $\underline{\underline{r}}$ is transformed into $\underline{\underline{r}}'$,

or as a change of co-ordinates with new origin at $-\underline{\underline{h}}$ and axes parallelly displaced.

(ii) Rotation, reflection.

$$\underline{\underline{r}}' = \underline{\underline{T}}_{\alpha}\underline{\underline{r}}, \qquad \underline{\underline{r}}' = \underline{\underline{S}}_{\ell}\,\underline{\underline{r}}$$

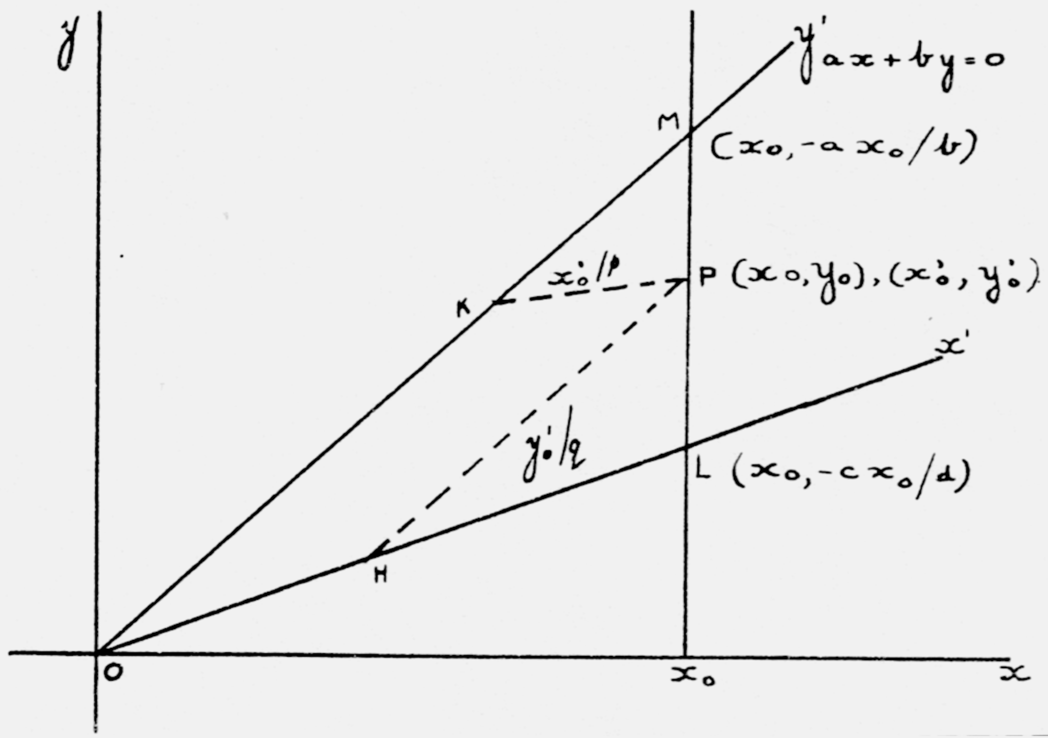
may be interpreted either as transformations of the plane

or as changes of axes by rotation through $-\alpha$ reflection in ℓ.

(iii) Affine change

$$r' = \underline{\underline{M}}\underline{\underline{r}}\ , \quad \underline{\underline{M}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \underline{\underline{M}} \text{ non-singular}$$

Regarded as a change of co-ordinate systems we have



L is $(x_o, -cx_o/d)$, M is $(x_o, -ax_o/b)$

$$x'_o = ax_o + by_o = p.*KP$$

$$y'_o = cx_o + dy_o = q.*HP$$

where p and q are constants:

$$p = (ad-bc)/\sqrt{(c^2+d^2)}$$

$$q = (ad-bc)/\sqrt{(a^2+b^2)}$$

The new axes are arbitrary lines $cx + dy = 0$,

$ax + by = 0$ through the origin and the new co-ordinates are proportional to constant multiples of the distances of the point from the new axes.

F. Combination of change of axes and transformation:

"similarity" of matrices.

Let us take an affine transformation of the plane in which the co-ordinates of P' and P are connected by

$$\underline{\underline{r}}' = \underline{\underline{M}}\,\underline{\underline{r}},$$

and a change of co-ordinates from (x, y), $\underline{\underline{r}}$, to (X, Y), $\underline{\underline{R}}$, where

$$\underline{\underline{R}} = \underline{\underline{C}}\,\underline{\underline{r}}, \text{ and } \underline{\underline{r}} = \underline{\underline{C}}^{-1}\,\underline{\underline{R}},$$

$\underline{\underline{C}}$ being non-singular.

In this new co-ordinate system, the co-ordinates (X', Y') of the point with original co-ordinates (x', y') are also given by $\underline{\underline{R}}' = \underline{\underline{C}}\,\underline{\underline{r}}'$

so that $\underline{\underline{R}}' = \underline{\underline{C}}\,\underline{\underline{r}}' = \underline{\underline{C}}\,\underline{\underline{M}}\,\underline{\underline{r}} = \underline{\underline{C}}\,\underline{\underline{M}}\,\underline{\underline{C}}^{-1}\,\underline{\underline{R}}$

Thus if, in the new co-ordinates, the transformation from P to P is represented by

$$\underline{\underline{R}}' = \underline{\underline{N}}\,\underline{\underline{R}},$$

then $$\underline{\underline{N}} = \underline{\underline{C}}\,\underline{\underline{M}}\,\underline{\underline{C}}^{-1}$$

$\underline{\underline{N}}$ is the <u>transform</u> of $\underline{\underline{M}}$ <u>by</u> $\underline{\underline{C}}$, and the matrices $\underline{\underline{M}}$ and $\underline{\underline{N}}$ are described as <u>similar</u>

<u>Theorem:</u> If $\underline{\underline{L}}$ is similar to $\underline{\underline{M}}$ and $\underline{\underline{M}}$ is similar to $\underline{\underline{N}}$ then $\underline{\underline{L}}$ is similar to $\underline{\underline{N}}$. "Similarity" is reflexive, symmetric, and transitive.

It may be noted that, using $\underline{\underline{C}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \underline{\underline{C}}^{-1}$, we find

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ is similar to } \begin{bmatrix} d & c \\ b & a \end{bmatrix}$$

The principal algebraic theorem for similar matrices states a condition which is sufficient but not necessary, although it is only in the "exceptional" cases that the converse theorem is not valid. In the usual applications made of the theorem only the sufficiency condition is required.

Theorem 2. If two matrices are similar then their characteristic functions are identical.

Proof : Let $\underline{\underline{N}} = \underline{\underline{C}}\ \underline{\underline{M}}\ \underline{\underline{C}}^{-1}$,

then
$$\det(x\underline{\underline{1}} - \underline{\underline{N}}) = \det(x\underline{\underline{1}} - \underline{\underline{C}}\ \underline{\underline{M}}\ \underline{\underline{C}}^{-1})$$
$$= \det(x\underline{\underline{C}}\ \underline{\underline{1}}\ \underline{\underline{C}}^{-1} - \underline{\underline{C}}\ \underline{\underline{M}}\ \underline{\underline{C}}^{-1})$$
$$= \det \underline{\underline{C}}\ (x\underline{\underline{1}} - \underline{\underline{M}})\ \underline{\underline{C}}^{-1}$$
$$= \det \underline{\underline{C}}.\ \det(x\underline{\underline{1}} - \underline{\underline{M}}).\ \det(\underline{\underline{C}}^{-1})$$
$$= \det(\underline{\underline{C}}\ \underline{\underline{C}}^{-1})\ \det(x\underline{\underline{1}} - \underline{\underline{M}})$$
$$= \det(x\underline{\underline{1}} - \underline{\underline{M}})$$

This theorem could be stated in the form:
"a sufficient condition for two matrices to have the same characteristic function is that they should be similar". The converse theorem: "a necessary condition for two matrices to have the same characteristic function is that they should be similar" is NOT TRUE. That is, there is NO THEOREM: if two matrices have the same characteristic function they are similar. For example, the set of matrices similar to $\alpha\underline{\underline{1}}$, namely,

$\underline{\underline{C}}\ (\alpha\ \underline{\underline{1}})\ \underline{\underline{C}}^{-1}$, consists of the single matrix $\alpha\underline{\underline{1}}$ itself. The characteristic function of $\alpha\underline{\underline{1}}$ is $(x - \alpha)^2$, which is the characteristic function of any matrix of the set $\left\{\begin{bmatrix} \alpha & u \\ 0 & \alpha \end{bmatrix}\right\}$

α fixed, u arbitrary. Such a matrix is similar to $\alpha\underline{\underline{1}}$ only if $u = 0$. We shall prove in fact (for 2 x 2 matrices, there is no corresponding simple result for larger matrices) that the only set of exceptional characteristic functions is $(x-\alpha)^2$ and the only exceptional matrices are those of the set $\{\alpha\underline{\underline{1}}\}$. We have however the following weaker theorem valid for all 2 x 2 matrices, and in fact for all square matrices:

Theorem 3. If $\underline{\underline{M}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\underline{\underline{N}} = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$

have the same characteristic function, i.e., if $\underline{\underline{M}}$ and $\underline{\underline{N}}$ are such that

$$a + d = a' + d'$$

and

$$ad - bc = a'd' - b'c'$$

then there exists a set of matrices $\{\underline{\underline{C}}\}$ such that

$$\underline{\underline{C}}\ \underline{\underline{M}} = \underline{\underline{N}}\ \underline{\underline{C}}$$

The theorem fails to be the complete converse because if $\underline{\underline{M}}$ and $\underline{\underline{N}}$ are to be similar, we require also that $\underline{\underline{C}}$ should be NON-SINGULAR. If for a given $\underline{\underline{M}}$ and $\underline{\underline{N}}$ all members of $\{\underline{\underline{C}}\}$ are singular, then $\underline{\underline{M}}$ and $\underline{\underline{N}}$ are not similar - there is no $\underline{\underline{C}}$ for which $\underline{\underline{C}}\,\underline{\underline{M}}\,\underline{\underline{C}}^{-1} = \underline{\underline{N}}$.

Proof: Assume that $\underline{\underline{C}} = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$

and that $\underline{\underline{C}}\ \underline{\underline{M}} = \underline{\underline{N}}\ \underline{\underline{C}}$; write

$$\underline{\underline{C}}\,\underline{\underline{M}} - \underline{\underline{N}}\,\underline{\underline{C}} = \underline{\underline{P}} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

Then $\underline{\underline{C}}\ \underline{\underline{M}} = \underline{\underline{N}}\ \underline{\underline{C}} \Longleftrightarrow \underline{\underline{P}} = \underline{\underline{O}}$. That is,

$$p \equiv hx + cy - b'z = 0$$
$$q \equiv bx - ky \quad -b't = 0$$
$$r \equiv -c'x \quad + kz + ct = 0$$
$$s \equiv \quad - c'y + bz - ht = 0,$$

where $h = a - a' = -(d-d')$

$$k = a - d' = a' - d,$$

and we have

$$bc - b'c' = ad - a'd' = -hk$$

We have the identities

$$bp + b's \equiv hq$$

$$c'p + cs \equiv -hr,$$

so that, if $h = O$,

$$(p = O \text{ and } s = 0 \text{ and } h \neq O)$$
$$\Longrightarrow (q = O \text{ and } r = O)$$

Similarly

$$(q = O \text{ and } r = O \text{ and } k \neq O)$$
$$\Longrightarrow (p = O \text{ and } s = O)$$

Thus for

$$h \neq 0,\ C_1 = \begin{bmatrix} (b'z - cy)/h & y \\ z & (bz - c'y)/h \end{bmatrix}$$

and for

$$k = 0,\ C_2 = \begin{bmatrix} x & (bx - b't)/k \\ (c'x - ct)/k & t \end{bmatrix}$$

both satisfy the condition for all pairs of values (y, z) and (x, t) respectively.

The matrix $\underline{\underline{C}}_1$ is singular if

$$bb' z^2 - (bc + b'c' + h^2) yz + c'y^2 = 0,$$

and all matrices of the set $\{\underline{\underline{C}}_1\}$ are singular if

$$bb' = 0, \quad cc' = 0,$$

and $\quad bc + h^2 = 0.$

We already have the relation

$$bc - b'c' + hk = 0,$$

so that $\quad bc = \frac{1}{2}h(h + k),$

$$b'c' = \frac{1}{2}h(h - k)$$

Since $h \neq 0$, the only two sets of solutions of the equations are

$$b = c = h + k = 0 \text{ which imply } a = d,$$

and $\quad b' = c' = h - k = 0$ which imply $a' = d'$,

for which respectively $\underline{\underline{M}} = a\underline{\underline{1}}$ and $\underline{\underline{N}} = a'\underline{\underline{1}}$.

The same result is obtained from the conditions that all matrices of the set $\{\underline{\underline{C}}_2\}$ are singular. Thus, provided either $h \neq 0$ or $k \neq 0$, the only sets of pairs of matrices M and N having the same characteristic function which are such that <u>every</u> matrix $\underline{\underline{C}}$ which satisfies the condition $\underline{\underline{C}}\ \underline{\underline{M}} = \underline{\underline{N}}\ \underline{\underline{C}}$ is singular are

$$\underline{\underline{M}} = a\underline{\underline{1}}, \quad N \neq a\underline{\underline{1}}$$

and $\quad \underline{\underline{M}} \neq a'\underline{\underline{1}}, \quad \underline{\underline{N}} = a'\underline{\underline{1}}$

We have now to examine the case $h = 0$, $k = 0$,

That is: $a = d = a' = d'$

and $\quad bc = b'c'$.

We have

$$p_o \equiv cy - b'z, \qquad s_o \equiv -c'y + bz$$
$$q_o \equiv bx - b't, \qquad r_o \equiv -c'x + ct$$

so that

$$bp_o + b's_o \equiv 0$$

and $\quad c'q_o + br_o \equiv 0$

Again therefore only two of the four equations are independent and the possible forms of the matrix C are:

$$b \neq 0, \quad C_3 = \begin{bmatrix} b't/b & y \\ c'y/b & t \end{bmatrix}$$

$$c \neq 0, \quad C_4 = \begin{bmatrix} x & b'z/c \\ z & c'x/c \end{bmatrix}$$

Always singular if and only if $b = c = 0$

and corresponding matrices for $b' \neq 0$, $c' \neq 0$, which are always singular if and only if $b = c = 0$. Thus again, the only cases in which all matrices of the set $\{C\}$ are singular are

$$\underline{\underline{M}} = a\underline{\underline{1}} \quad , \quad \underline{\underline{N}} \neq a\underline{\underline{1}},$$

$$\underline{\underline{M}} \neq a'\underline{\underline{1}} \quad , \quad \underline{\underline{N}} = a'\underline{\underline{1}}$$

We have now exhausted all the possibilities.

The exceptional cases all occur when the characteristic function is the square of a linear form, that is,

when $(a + d)^2 = 4(ad - bc)$,

so that the full theorem on similarity and the characteristic functions for 2 x 2 matrices may be stated thus:

Theorem 4 : (i) The set of matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

both non-singular and singular, for which either

$$(a - d)^2 + 4bc \neq 0$$

or $(a - d)^2 + 4bc = 0$ and not both $b = 0$ and $c = 0$,

can be split into a set $\{\Sigma\}$ of mutually exclusive subsets Σ by the equivalent properties:

two matrices belong to the same subset Σ if and only if (a) they are similar, or (b) they have the same characteristic function.

(ii) The set of exceptional matrices, that is, those for which $b = c = a - d = 0$, namely, the set $\{a\,\underline{\underline{1}}\}$, including $\underline{\underline{0}}$, is such that each matrix, although it has the same characteristic function as the matrices in some subset Σ, is similar only to itself.

It is necessary to emphasize that the theorem in this form is peculiar to 2 x 2 matrices, the full statement for larger matrices is much more complicated. It would never of course be expected that students would "learn" the proofs of the converse theorem or even the theorem itself, but the sufficiency theorem and its method of proof are

of considerable importance. The value of the converse theorem lies in the way it illustrates the vagaries of the solutions of linear equations in relation to conditions which are sufficient but not necessary. Another facet of the same algebra is the determination of the matrices which commute with a given matrix, i.e., of the set $\underline{\underline{C}}$ of matrices for which, for a given matrix $\underline{\underline{M}}$, $\underline{\underline{M}}\ \underline{\underline{C}} = \underline{\underline{C}}\ \underline{\underline{M}}$

NOTE:

For rotations and reflections, since

$$\underline{\underline{T}}^T = \underline{\underline{T}}^{-1} , \quad \underline{\underline{S}}^T = \underline{\underline{S}}^{-1} ,$$

the relation of similarity takes the form

$$\underline{\underline{N}} = \underline{\underline{S}}\ \underline{\underline{M}}\ \underline{\underline{S}}^T , \quad \underline{\underline{N}} = \underline{\underline{T}}\ \underline{\underline{M}}\ \underline{\underline{T}}^T$$

H. Reduction of quadratic forms - Conics

General form: $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

First step: shift of origin to produce in general

$$ax^2 + 2hxy + by^2 + \Delta/C = 0$$

Cases $C = 0$, $\Delta = 0$.

Second step: Reduction of $ax^2 + 2hxy + by^2 = 1$,

$$\text{i.e.,} \quad [x, y] \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \underline{\underline{r}}^T\ \underline{\underline{C}}\ \underline{\underline{r}} = 1,$$

by rotation of axes.

$$\text{Take} \quad \underline{\underline{r}}' = \underline{\underline{T}}_{-\theta}\ \underline{\underline{r}} ,$$

$$\text{i.e.} \quad \underline{\underline{r}} = \underline{\underline{T}}_{\theta}\ \underline{\underline{r}}' ,$$

and suppose in the new co-ordinates $ax^2 + 2hxy + by^2 = 1$

becomes $\alpha x'^2 + \beta y'^2 = 1$, i.e. $\underline{\underline{r}}'^T \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \underline{\underline{r}}' \equiv \underline{\underline{r}}'^T \underline{\underline{D}}\underline{\underline{r}} \equiv 1$

We have then

$$\underline{\underline{r}}\ \underline{\underline{C}}\ \underline{\underline{r}} = (\underline{\underline{T}}_\theta \underline{\underline{r}}')^T\ \underline{\underline{C}}\ \underline{\underline{T}}_\theta\ \underline{\underline{r}}' = \underline{\underline{r}}'^T (\underline{\underline{T}}_\theta^T \underline{\underline{C}}\ \underline{\underline{T}}_\theta)\ \underline{\underline{r}}'$$
$$= \underline{\underline{r}}'^T\ \underline{\underline{D}}\ \underline{\underline{r}}'$$

So that $\quad \underline{\underline{T}}_\theta^T\ \underline{\underline{C}}\ \underline{\underline{T}}_\theta = \underline{\underline{D}}$

Thus $\underline{\underline{C}}$ and $\underline{\underline{D}}$ are similar, so that

$$a + b = \alpha + \beta,$$

$$ab - h^2 = \alpha\beta,$$

specifying the form of the conic.

α, β are the eigenvalues of $\underline{\underline{C}}$, the eigenvectors lie along the principal axes of the conic.

If we want the angle θ through which the axes are turned we have to equate to zero the element 12 in the matrix $\underline{\underline{T\theta}}^T \underline{\underline{C}}\ T\theta$. We find

$\tan\ 2\theta = 2h/(a-b)$.

The only work on conics need consist of the identification with the curves above of the curves given by the focus-directrix definition, and the shapes of these curves.

I. Note on Complex Numbers as Matrices

(to supplement the usual treatment of the Argand diagram.)

$z = x + iy \quad = r(\cos\theta + i\sin\theta)$

$$\underline{\underline{z}} = \begin{bmatrix} x \\ y \end{bmatrix} = r\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\underline{\underline{\underline{Z}}} = \begin{bmatrix} x & -y \\ y & x \end{bmatrix} = r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$z'' = z + z' \longleftrightarrow \underline{\underline{z}}'' = \underline{\underline{z}} + \underline{\underline{z}}' \longleftrightarrow \underline{\underline{\underline{Z}}}'' = \underline{\underline{\underline{Z}}} + \underline{\underline{\underline{Z}}}'$$

$$z'' = zz' \longleftrightarrow \underline{\underline{z}}'' = \underline{\underline{\underline{Z}}}\underline{\underline{z}}' \longleftrightarrow \underline{\underline{\underline{Z}}}'' = \underline{\underline{\underline{Z}}}\underline{\underline{\underline{Z}}}'$$

$\frac{1}{z}$ corresponds to $\underline{\underline{\underline{Z}}}^{-1}$. $\underline{\underline{\underline{Z}}}\bar{\underline{\underline{\underline{Z}}}} = (x^2 + y^2)\, \underline{1}$.

If α is a point, $\underline{\underline{\alpha}}$, $= \begin{bmatrix} a \\ b \end{bmatrix}$, on the Argand diagram and

$\lambda = \rho(\cos\theta + i\sin\theta)$,

Then $\lambda\alpha$ corresponds to $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$,

i.e., the point obtained by rotation and multiplication by ρ in the usual way.

J. Groups

(i) Introduction

The sets of rotations through 180°; 90°, 180°, 270°, 240°; ... $\{(360\ r/n)^\circ\}$ about 0 and effect of these transformations on corresponding regular polygons with centre at 0.
Combination of any of these with a reflection in line through 0 and vertex or mid-point of side of the polygon.

(ii) Definition of a group:

A group is a set, say $G = \{A, B, C, \ldots\}$

together with an operation, $*$, for combining two members to produce a third, which has the properties:

1. $A \in G$ and $B \in G \Longrightarrow A*B \in G$ (closed)

2. $(A * B) * C = A * (B * C)$ (associative)

3. There is $\underline{\underline{1}}$, $\underline{\underline{1}} \in G$, such that

$$A * \underline{\underline{1}} = \underline{\underline{1}} * A = A$$

4. For every A there is $A^{-1} \in G$ such that

$$A * A^{-1} = A^{-1} * A = \underline{\underline{1}}$$

(In general $A * B \neq B * A$).

Change notation: $A * B$ in future written as AB.

<u>Theorem:</u> $(AB)^{-1} = B^{-1}A^{-1}$

Various sets of rotations (and all rotations) about 0 form (commutative) groups.

Reflections together with rotations (with fixed 0) form a (non-commutative) group.

(iii) <u>Special finite groups that might be used as examples.</u>

a. Symmetric group on three elements (A_3).

a1. <u>Symmetries of equilateral triangle PQR.</u>

(For convenience of typing put S, T, $\underline{\underline{1}}$ for $\mathcal{S}$ $\mathcal{T}$, $\mathcal{I}$.)

T: rotation about centroid through 120^o

S: reflection in altitude through R.

The various positions of the triangle after combinations of these transforms are:

(TS is "S followed by T")

$\underline{\underline{1}}$	T	T^2	S	
R	Q	P	R	
P Q	R P	Q R	Q P	

TS	T^2S	ST	ST^2
P	Q	Q	P
R Q	P R	P R	R Q

STS	ST^2S	TST	TST^2
P	Q	R	Q
Q R	R P	Q P	P R

The following pairs have identical effects on the triangle:
S and TST, T^2 and STS, T and ST^2S, TS and ST^2 etc. That is, among the members of the set of operations we have the relations
$T^3 = \underline{\underline{1}}$, $S^2 = \underline{\underline{1}}$, S = TST, T^2 = STS, T^2 = ST S etc.
But $T = ST^2S \Longleftrightarrow TS = ST^2S^2 = ST^2 \Longleftrightarrow TST = ST = S$ etc.
so that the only distinct relations are

$$\underline{\underline{1}} = S^2 = T^3 = (TS)^2$$

and the only distinct operators are

$$\underline{\underline{1}},\ S,\ T,\ T^2,\ ST,\ TS$$

a2. Permutations of three symbols (a, b, c)

Take

U: the operation of interchanging symbols in first and second places

V: the operation of interchanging symbols in first and third places

$$U^2 = \underline{\underline{1}},\quad V^2 = \underline{\underline{1}}$$

$$U(a,b,c) = (b,a,c),\quad V(a,b,c) = (c,b,a)$$

$$VU(a,b,c) = (c,a,b),\quad UV(a,b,c) = (b,c,a)$$

$$UVU(a,b,c) = (a,c,b),\quad VUV(a,b,c) = (a,c,b)$$

We have UVU = VUV, and have formed all six permutations.

The relations are

$$U^2 = \underline{\underline{1}},\ V^2 = \underline{\underline{1}},\ UVU = VUV \Longleftrightarrow (UV)^3 = \underline{\underline{1}} = (VU)^3$$

We can identify with (a1.) by taking

$$U = S,\quad V = TS$$

$(UVU = STSS = ST = T^2S = TSSTS = VUV)$

(The identity of these two groups is in fact clear from their original definitions).

a3. A_3 as a certain set of rational functions

Take $x \in \{\text{Reals}\} - \{0, 1\}$.

Define $f(x) = 1/x$, $g(x) = 1 - x$

$$f(g(x)) = 1/(1-x),\quad g(f(x)) = 1 - 1/x = (x-1)/x$$

$$g(f(g(x))) = 1 - 1/(1-x) = x/(x-1),\quad f(g(f(x))) = x/(x-1)$$

Treating f and g as operators, we have

$$f^2 = g^2 = 1, \quad fgf = gfg$$

They generate a group identical with (a1.) and (a2.).

b. Groups $\{1, \ldots, p-1\}$ under multiplication, $\{0 \ldots . p-1\}$ under addition, p prime, for remainders modulo p.

$\{1, \ -1, \ i, \ -i\}$ under multiplication

c. <u>The group of rotations, reflections, displacements.</u>

(a) <u>Origin fixed</u>

The operators T_α form a commutative group, with rules of combination $T_\alpha T_\beta = T_{\alpha+\beta} = T_\beta T_\alpha$

$$T_{\alpha+2r\pi} = T_\alpha$$

The operators S_a do not form a group, since the resultant of two reflections is not a reflection, but S_a and T_α together form a group.

(b) <u>Translations form a commutative group:</u>

(c) <u>Congruence</u>

$$D_{AB} = S_\ell S_m \quad \text{where } \ell \perp AB, \ m \perp AB$$

and distance between ℓ and $m = \frac{1}{2} * AB$

We may replace any translation by reflections in a pair of parallel lines, one of which could be made to pass through an assigned point, and, using the relations $S_a S_b = T_{\angle *ba}$ and $D_{AB} S_a = S_a D_{A'B'}$ we may reduce any system of translations, reflections and rotations to a transformation consisting of some combination of

(α) a rotation round any given point 0,

(β) a reflection in a line through 0, and

(γ) a reflection in a line not through 0.

namely, (α) with (γ) or (β) with (γ) or (α) or (β) or (γ) alone

d. <u>Groups defined by matrices under multiplication</u>

(a) Reflection and rotation matrices generate the same groups as the corresponding geometric operators. E.g. the group A_3 is generated by

$$\underline{\underline{T}}_{120^\circ} = \underline{\underline{T}} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{bmatrix}, \quad T^3 = \underline{\underline{1}}$$

$$\underline{\underline{S}}_{0y} \equiv \underline{\underline{S}} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad S^2 = \underline{\underline{1}} \text{ etc.}$$

The group of symmetries of the square:

$$\underline{\underline{S}}_{0y} \equiv \underline{\underline{S}} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{\underline{S}}_{x=y} \equiv \underline{\underline{S}}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\underline{\underline{S}}^2 = \underline{\underline{S}}'^2 = 1 \,. \; (\underline{\underline{S}}\underline{\underline{S}}')^2 = (\underline{\underline{S}}'\underline{\underline{S}})^2 = -\underline{\underline{1}} \,.$$

(b) Groups generated by matrices over modular fields.

E.G., in the field of remainders on division by 2;

$$\underline{\underline{M}} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \underline{\underline{M}}^2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad \underline{\underline{M}}^3 = 1$$

$$\underline{\underline{N}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \underline{\underline{N}}^2 = 1, \; (\underline{\underline{M}}\underline{\underline{N}})^2 = (\underline{\underline{N}}\underline{\underline{M}})^2 = \underline{\underline{1}} \text{ (Group } A_3\text{)}$$

(c) The matrices of the set $\{\underline{\underline{C}}\}$ such that for a given matrix $\underline{\underline{M}}$, $\underline{\underline{C}}\,\underline{\underline{M}}\,\underline{\underline{C}}^{-1} = \underline{\underline{M}}$. (Do the matrices for which $\underline{\underline{C}}\,\underline{\underline{M}} = \underline{\underline{M}}\,\underline{\underline{C}}$ form a group?)

(c) Analytical geometry of three dimensions

For the first two or three years this topic will be examined on the assumption that it has been treated in the way described in the notes on item II. 14. The intersection of three planes and the solution of three simultaneous linear equations may be treated as at present (see, for example, Report on Leaving Certificate 1962, Mathematics Honours I, question 1). 3 x 3 matrices and their application to three dimensional geometry will not be examined at present. Ample notice will be given of any intention to set questions on this treatment of the subject.

5. ELEMENTARY DYNAMICS OF A PARTICLE

Throughout the work on this topic the emphasis should be on the use of the calculus to express the relations of mechanics, and not on the solution of complicated problems. It is not intended that a large number of formulae should be committed to memory but rather that the student should be able to derive any result from first principles whenever it is required.

Rectilinear motion of a particle may be described by a functional relation $x = f(t)$ where x is distance measured from an origin and t is time from a given instant. Velocity and acceleration will be defined as differential co-efficients $\dot{x}$ and $\ddot{x}$ where the dots indicate differentiation with respect to t.

The kinematical formulae

$$\ddot{x} = a, \ \dot{x} = u + at, \ x = ut + \tfrac{1}{2} at^2$$

for uniformly accelerated motion will be derived.

The classical statement of Newton's Laws of motion (excluding the third law) should be given. By choice of units the Newtonian formula F = ma follows. For motion of a particle in a straight line this becomes

$$m \frac{d^2x}{dt^2} = F$$

where F is the force acting on the particle at time t in the direction in which x increases. If F is known at each instant of the motion this is a differential equation to determine the motion.

Setting $v = \dot{x}$ the equation may be written

$$m v \frac{dv}{dx} = F$$

or

$$\frac{d}{dx}\left(\frac{1}{2} mv^2\right) = F$$

Then, integrating from x_1 to x_2 with corresponding velocity values v_1, v_2 we get

$$\tfrac{1}{2} mv_2^2 - \tfrac{1}{2} mv_1^2 = \int_{x_1}^{x_2} F \, dx$$

On the left we have the increment in the quantity $\frac{1}{2} mv^2$, which is called the kinetic energy of the particle of mass m moving with velocity v; on the right we have the work done by the force acting on the particle during its displacement.

Thus

> Increase in kinetic energy = work done by the external force acting on the particle.

In cases where the force acting at a position x depends on this position only, then F is a function of x, and

$$V = \int_a^x F \, dx$$

is a function of x also. V is called the potential energy of the particle at x. Then

$$\int_{x1}^{x2} F \, dx = V_1 - V_2$$

and the energy equation may be written

$$\tfrac{1}{2} mv_1^2 + V_1 = \tfrac{1}{2} mv_2^2 + V_2$$

So the quantity

$$\tfrac{1}{2} mv^2 + V$$

is constant throughout the motion. This is a statement of the conservation of (mechanical) energy - the sum of kinetic and potential energies is constant. Of course it is a mathematical theorem derived by integration of the equation of motion - and applies only in the 'conservative' field of force considered.

The important cases in elementary work are (i) the case when F is constant (motion under gravity), (ii) the case when F is proportional to x (motion near an equilibrium position) leading in particular to simple harmonic motion defined by the equation

$$\frac{d^2x}{dt^2} = -n^2x ,$$

with solutions

$$x = a \sin (nt + b)$$

$$x = a \cos (nt + b)$$

$$x = A \cos nt \pm B \sin nt$$

Discussion of resisted motion should be restricted to the case of a particle moving vertically under gravity and subject to a resistance proportional to its velocity. The equation of motion will then be

$$\frac{dv}{dt} = -kv - g \quad \text{or} \quad \frac{dv}{dt} = -kv + g$$

and the solutions are easily found.

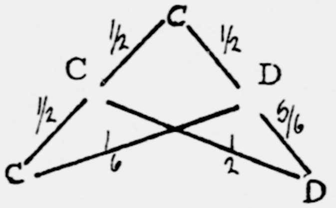
Motion in two dimensions should deal only with the parabolic motion of a projectile under gravity. Find the height of flight, and the range on a horizontal plane, for given conditions of projection, and the maximum height, and maximum range for a given speed of projection.

It is suggested that the work on dynamics should not be treated as a separate unit, but that, as far as possible the applications should be made whenever appropriate to illustrate the use and value of the calculus. It may be recalled that the subject was developed originally in just this context.

THEORY OF PROBABILITY

The treatment is to be the same as that described in the Notes to Level II Item 19, but the following possible type of application of matrix algebra to the representation of certain stochastic processes could be introduced as an example of the use of matrix algebra.

A succession of operations consists of either tossing a coin or throwing a die. If the coin comes up heads or the die shows "6", the next operation is tossing the coin, otherwise, throwing the die. Represent the operations by C, D, then the succession from C is



(probabilities) $\left(\frac{1}{4} + \frac{1}{12}\right)$, $\left(\frac{1}{4} + \frac{5}{12}\right)$

Thus, if at any stage the chance that the operation is C is p, and that it is D is q(= 1-p), then, at the next stage

$$\begin{bmatrix} p' \\ q' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{6} \\ \frac{1}{2} & \frac{5}{6} \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \left\{ \underline{\underline{1}} - \begin{bmatrix} \frac{1}{2} & -\frac{1}{6} \\ -\frac{1}{2} & \frac{1}{6} \end{bmatrix} \right\} \begin{bmatrix} p \\ q \end{bmatrix}$$

Matrices $M = \begin{bmatrix} a & -b \\ -a & b \end{bmatrix}$ are such that $\underline{\underline{M}}^2 = (a+b)\underline{\underline{M}}$ and

$$(\underline{\underline{1}} - \underline{\underline{M}}) = \underline{\underline{1}} - \frac{1}{(a+b)} \underline{\underline{M}} + \frac{(1-a-b)^n}{a+b} \underline{\underline{M}} \longrightarrow \text{(etc.) as } n \longrightarrow \infty \text{ ?}$$

APPENDIX

The relation of the Second Level (2F) Syllabus to the First Level Syllabus

The Second Level (2F) Syllabus is wholly contained in the First Level Syllabus. The two courses preferably should be taught separately, so that full advantage can be taken of the First Level student's quicker perception and higher rate of working, but in schools where very few students study Mathematics at First Level it should be practicable to teach First Level and Second Level (2F) students in the same class for part of each week (say not more than five periods) throughout the two years.

The First Level Syllabus may be thought of as constructed in three tiers:

Tier 1:	the part which forms the Second Level (2F) Syllabus,
Tier 2:	extensions of topics in the Second Level (2F) Syllabus,
Tier 3:	topics with no significant counterpart in the Second Level (2F) Syllabus.

For example, in item I. 1 of the First Level Syllabus, item I. 1(d) includes (Tier 1) the whole of item II. 5 in the Second Level Syllabus, and then proceeds (Tier 2) to a discussion of a convergence to a greater depth than is required for Second Level. On the other hand, item I. 1(c), complex numbers, belongs to Tier 3 and has no counterpart in the Second Level Syllabus.

An advantage of this arrangement is that, if circumstances are such as to necessitate the teaching of students at First and Second (2F) Levels together for part of the time, then, while the Second Level (2F) students are spending time consolidating the work done by the combined class, the First Level students can proceed to Tier 2 or to Tier 3. Tier 3 also can be made to act as a flywheel, running continuously for two or three of the periods each week independently of the topics that are reqiired to be taught in the combined class.

The principal items in Tier 3 are I. 1(e) and I. 4(b).

One of the items which is effectively the same for First Level and Second Level (2F) is item II. 19 = item I. 6. This item could be introduced at almost anytime after the binomial theorem (item II. 18) which in turn could be introduced at almost any stage. Another is three-dimensional geometry (for the first few years of operation of this Syllabus) Item I. 4(c) = item II. 14.

If it is deemed necessary to cater for students taking SCIENCE at Second Level (2F) or First Level, then the Calculus should be introduced as soon as practicable; the order of the items in the Mathematics Second Level (2F) has been chosen with this object in view.

For the convenience of those whose circumstances do not allow First Level students to be taught separately all the time, the following table of relations between the Second Level (2F) and First Level Syllabuses has been prepared.:

II F Item No.	Tier 1 Item No.	I Tier 2 Item No.	Tier 3 Item No.
1 a	1 b	1 a	
1 b	1 c	1 c	1 e
2 a, b, c	3 a		
3*	-		
4 a to e	4 a		4 b
5 a to f	1 d	1 d	
6 a	3 b		
6 b to i	3 c	3 c	
7 a to d	2 a		
7 e, f	4 a		
8 a, b, c	(4b)		4 b
9 a to g	3 c	3 c	
10 a to d	3 d	3d, 3g	
11 a to c	3 c, e	3 e	
12 a to h	3f	3f	
13 a to e	3f	3f	
14 a to d	4 c		(4c)
15 a to e	5	5	
16 a to c	3f		
17a	2a	2a	
17 b to d	2b	2b	
18 a to c	2c	2c, 3h	
19 a to f	6		

* This item is in the Advanced Level Course for the School Certificate but not in the ordinary-with-Credit Level Course. I. 4 b could be started during the teach[ing] of II. 3.

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MATHEMATICS SYLLABUS - FIRST LEVEL

FORMS V. & VI.

Bibliography for use with the First Level Mathematics Syllabus, "Introduction to Matrix Algebra" - see p. 28 of the Syllabus.

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The following list of books on Matrices, helpful to teachers, replaces that issued on 7th September, 1966.

P.J. DAVIS	Mathematics of Matrices (Blaisdell)
P.O. REDGRAVE	Teachers' Manual (to Davis' book) (Blaisdell)
D. PEDOE	A Geometric Introduction to Linear Algebra (Wiley)
J. SCHWARTZ	Introduction to Matrices and Vectors (McGraw-Hill)
G. STEPHENSON	An Introduction to Matrices, Sets and Groups (Longmans)
SCHOOL MATHEMATICS STUDY GROUP	Introduction to Matrix Algebra (Yale University Press)
SCHOOL MATHEMATICS PROJECT	Book T4 (Cambridge University Press)
J. CORONEOS & J.A. LYNCH	A Higher School Certificate Course in Mathematics - Form 5; Level I Form 6; Level I
H. MULHALL & W.B. SMITH-WHITE	A New Mathematics for Senior Schools Advanced Supplement. Part 2. (Angus & Robertson.)

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(Distribution of this List approved by the Board on 19th October, 1966)