Question 1 (15 Marks) **Marks**

(a) Find:

(i) $\int \frac{e^x}{\sqrt{e^{2x}-1}}\,dx$ **2**

(ii) $\int \frac{1}{x^2-5x+6}\,dx$ **2**

(iii) $\int \frac{d\theta}{2+\cos\theta}$ **3**

(b) Evaluate: $\int_{-1}^{1} \frac{x}{x^2+2x+5}\,dx$ **4**

(c) If $I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + 2\sin x}\,dx$ and $J = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x + 2\sin x}\,dx$,

(i) Show that $2I - J = \ln 2$. **1**

(ii) Evaluate $I + 2J$. **1**

(iii) Hence, find the exact values of *I* and *J*. **2**

Question 2 (15 Marks) START A NEW PAGE

(a) Plot neatly on an Argand diagram the points *A*, *B* and *C* corresponding to the complex numbers w, w^2 and $w\overline{w}$ respectively where $w=\sqrt{3}+i$. **3**

(b) Let $z = x + iy$ be a complex number satisfying the inequality **4**

$$z\overline{z}+(1-2i)z+(1+2i)\overline{z}\le 4 \qquad \text{where } x \text{ and } y \text{ are real.}$$

Sketch the locus of *z* on an Argand diagram.

(c) (i) Solve the equation for *w*: **2**

$$w^2 = -11-60i.$$

Write your answer in the form $w = x + yi$, where x and $y \in \mathrm{R}$

(ii) Hence, or otherwise, solve the equation: **3**

$$z^2-(1+4i)z-(1-17i)=0$$

(d) Five girls and three boys are seated at random around a circular table. **3**
What is the probability that at least two boys are sitting next to each other?

Question 3 (15 Marks) START A NEW PAGE **Marks**

(a) *ABCD* is a cyclic quadrilateral. Chords *BE* and *DF* bisect $\angle ABC$ and $\angle ADC$ respectively.

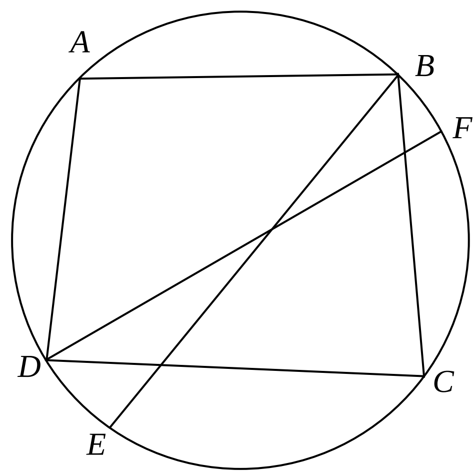


Diagram not to scale

Copy the diagram and prove that *EF* is a diameter of the circle. **3**

(b) (i) Show whether the function $f(x) = 2|x-1| - |x| + 2|x+1|$ is even, odd or neither, giving reasons. **2**

(ii) Sketch the graph of the function $f(x) = 2|x-1| - |x| + 2|x+1|$, clearly showing all intercepts with the coordinate axes and critical points. Label all branches with the relevant equations. **3**

(c) $P(x_1, y_1)$ is a point on the rectangular hyperbola $xy = 9$.

(i) Show that the Cartesian equation of the tangent at *P* is $y_1 x + x_1 y = 18$. **2**

(ii) Hence, or otherwise, derive the equation of the chord of contact from an external point $T(x_0, y_0)$ to the hyperbola $xy = 9$. **2**

(iii) Prove that the chord of contact is a focal chord when *T* is a point on the directrix. **3**

Question 4 (15 Marks) START A NEW PAGE **Marks**

(a) (i) Find all stationary points for the curve $y^2 = x(3-x)^2$. **3**

(ii) Sketch the curve $y^2 = x(3-x)^2$, showing all stationary points and the intercepts with the coordinate axes. **3**

(b) A particle of mass 2kg is projected vertically upwards with a velocity of U ms^{-1} in a medium which exerts a resistive force of $\frac{v}{10}$ Newtons.

(i) Show that the maximum height H metres reached by the particle is given by: **3**

$$H = 20U + 4000\ln\left(\frac{200}{200+U}\right) \qquad \text{(take } g = 10 \text{ ms}^{-2}\text{)}$$

(ii) Find the time taken for the particle to reach the maximum height H. **3**

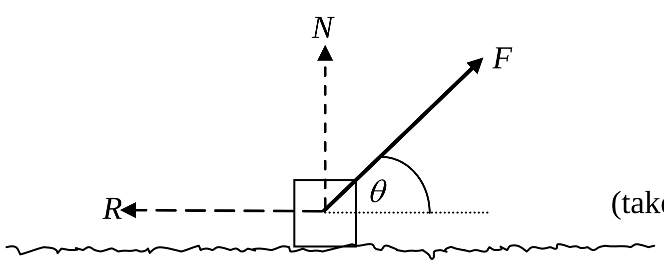
(iii) If $U = 400$, show that the average speed during the ascent is: **3**

$$200\left(\frac{2}{\ln 3} - 1\right) \text{ ms}^{-1} .$$

Question 5 (15 Marks) START A NEW PAGE

(a) A block of mass 5 kg is to be moved along a rough horizontal surface by a force (F Newtons) inclined at an angle of θ with the direction of motion where $0 \le \theta \le \frac{\pi}{2}$.

Diagram not to scale



(take $g = 10$ ms^{-2})

The motion is resisted by a frictional force (R Newtons) which is proportional to the normal reaction force (N Newtons) exerted on the block by the surface, such that $R = 0{\cdot}2N$.

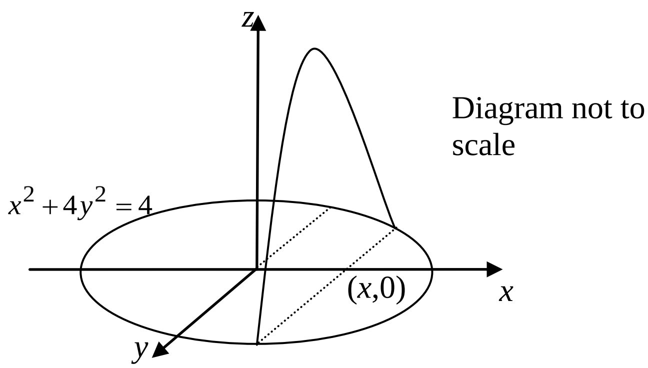
(i) Show that $F = \dfrac{50}{5\cos\theta + \sin\theta}$ Newtons, when the block is about to move. **3**

(ii) Calculate the minimum value of F needed to overcome the frictional resistance between the block and the surface. **3**

Question 5 continued over page

(b) (i) A parabola has the equation $x^2 = 4ay$. Show that the area bounded by this parabola and the focal chord perpendicular to the axis is equal to $\frac{8a^2}{3}$ units^2. **3**

(ii) A solid has an elliptical base whose equation is $x^2 + 4y^2 = 4$ and each cross-section perpendicular to the major axis of the base is a parabola with its focus on the major axis.



(α) Show that the area of the parabolic cross-section, x units from the origin, is given by the formula

$$A(x) = \frac{4 - x^2}{6}$$ **3**

(β) Hence, find the volume of the resultant solid. **3**

Question 6 (15 Marks) START A NEW PAGE

(a) The points $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ lie on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{where} \quad 0 < \theta < \phi \text{ and } a > b.$$

The points $P'(a\cos\theta, a\sin\theta)$ and $Q'(a\cos\phi, a\sin\phi)$ lie on the auxiliary circle and subtend a right angle at the origin.

(i) Draw a neat sketch of the above information showing the relative positions of the points P, Q, P' and Q'. **2**

(ii) Express the coordinates of Q in terms of θ. **1**

(iii) The tangents at P and Q meet in point R. Find the coordinates of R in terms of θ. **4**

(iv) Show that R lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$ **1**

Question 6 continued over page

(b) (i) If $\tan(x)\tan(\theta - x) = k$ prove that: **4**

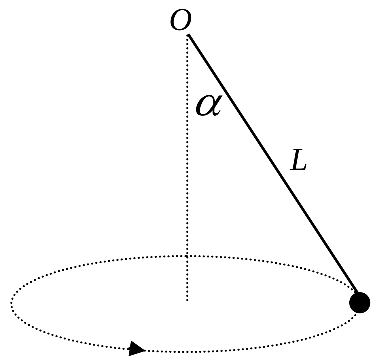
$$\frac{1+k}{1-k} = \frac{\cos(2x-\theta)}{\cos\theta}$$

(ii) Hence, or otherwise, solve the equation for all *x*. **3**

$$\tan x \tan\left(\frac{\pi}{3} - x\right) = 2 + \sqrt{3}$$

Question 7 (15 Marks) START A NEW PAGE

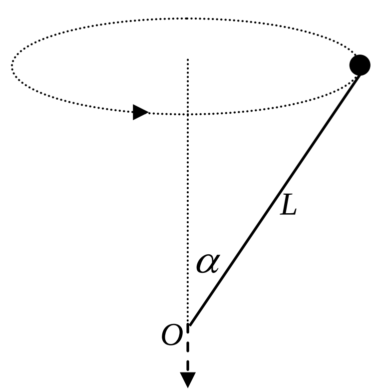
(a) A particle of mass *m* kg is fastened to one end of a light inextensible string of length *L* metres and the other end is attached to a fixed point *O*. The particle rotates with a uniform angular velocity ω rad/s about a vertical line through *O*.



(i) Show that if α is the angle of inclination of the string to the downward vertical, then $\alpha = \cos^{-1}\left(\frac{g}{L\omega^2}\right)$. **4**

(ii) Explain why steady circular motion is only possible when $\omega^2 > \frac{g}{L}$. **2**

(iii) The point *O* is now made to descend with an acceleration of f ms^{-2}, whilst the particle continues to rotate with uniform angular velocity ω.

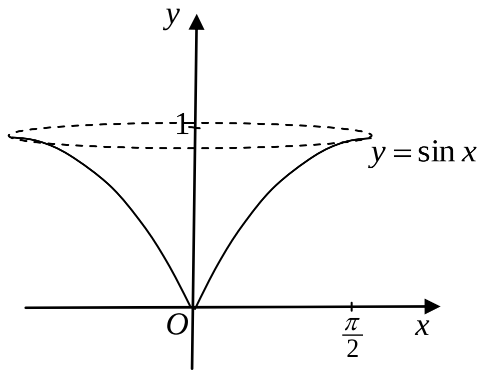


Find the value of f when the string again makes a constant angle of α with the upward vertical. **3**

Question 7 continued over page

Question 7 continued **Marks**

(b) The area between the curve $y = \sin x$, from $x = 0$ to $x = \frac{\pi}{2}$, the y-axis and the line $y = 1$ is rotated about the y-axis.



(i) Show that the volume of the solid formed can be found by using the formula **3**

$$V = \pi \int_0^{\frac{\pi}{2}} x^2 \cos x \, dx$$

(ii) Hence, calculate the volume of the solid. **3**

Question 8 (15 Marks) START A NEW PAGE

(a) The total number of different groups with 4 members which can be chosen from a group of n people is five times as many as the total number of different groups with 3 members which can be chosen from a group of $n-2$ people. **3**

Find all possible values of n.

(b) Prove that $\tan^{-1}(5) + \tan^{-1}(3) + \tan^{-1}\left(\frac{4}{7}\right) = \pi$ **4**

(c) A curve, defined by the equation $x^2 + 2xy + y^5 = 4$, has a horizontal tangent at the point $P(X, Y)$.

(i) Show that X is a root to the equation $x^5 + x^2 + 4 = 0$. **3**

(ii) Show that there is root to the equation $x^5 + x^2 + 4 = 0$ between -2 and -1. **1**

(iii) With the use of a graph, or otherwise, show that X is the only real root to the equation $x^5 + x^2 + 4 = 0$. **4**

End of Examination