T. Lee

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TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2001

MATHEMATICS

EXTENSION II

Time Allowed – 3 Hours
(Plus 5 minutes reading time)

All questions may be attempted

All questions are of equal value

In every question, show all necessary working

Marks may not be awarded for careless or badly arranged work

Standard integral tables are printed at the end of the examination paper and may be removed for your convenience. Approved silent calculators may be used.

The answers to all questions are to be returned in separate bundles clearly labelled Question 1, Question 2 , etc. Each bundle must show your candidate number.

Question 1 **Marks**

(a) Find $\int \frac{x^3}{x-2}dx$. **2**

(b) Evaluate $\int_{-2}^{2}(x\cos^2 x - 100x^5 + 2)dx$. **2**

(c) (i) Express $\frac{3x+7}{(x+1)(x+2)(x+3)}$ in partial fractions. **3**

(ii) Prove that $\int_0^1 \frac{(3x+7)dx}{(x+1)(x+2)(x+3)} = \ln 2$. **2**

(d) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx$ where n is a non negative integer.

(i) Show that $I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x dx$ where $n \geq 2$. **3**

(ii) Deduce that $I_n = \frac{n-1}{n} I_{n-2}$ where $n \geq 2$. **2**

(iii) Evaluate I_4. **1**

Question 2 (Start a new page)

(a) The hyperbola h has equation $x^2 - y^2 = 4$.

(i) Find the foci, asymptotes and vertices.. **3**

(ii) Prove that the equation of the normal to h at $P(4, 2\sqrt{3})$ is $2\sqrt{3}y + 3x = 24$. **3**

(iii) Find the equation of the circle that is tangent to h at P and $Q(-4, 2\sqrt{3})$. **3**

(iv) Using complex numbers or otherwise show that $x^2 - y^2 = 4$ transforms to $xy = 2$ if we choose the asymptotes as the x and y axes with the curve in the first and third quadrants. **3**

(v) Find the foci of $xy = 2$. **1**

(b) Of three cards, one is green on both faces, one white on both faces, whilst the third is green on one side and white on the other. They are placed in a hat, one is withdrawn and placed on a table. If the visible face is green, what is the probability that the other face is also green? **2**

Question 3 (Start a new page) **Marks**

(a) Define <u>modulus</u> and <u>conjugate</u> of a complex number $z = x + iy$ (x, y real). Prove that: **1**

(i) $|z|^2 = z\bar{z}$ and that, for any two complex numbers z_1 and z_2. **2**

(ii) $(\overline{z_1 z_2}) = (\bar{z_1})\,(\bar{z_2})$. **2**

(iii) Deduce that $|z_1 z_2| = |z_1|\,|z_2|$. **2**

(b) Draw neat, labeled sketches (not on graph paper) to indicate each of the subsets of the Argand diagram described below (all necessary detail such as intercepts to be shown)

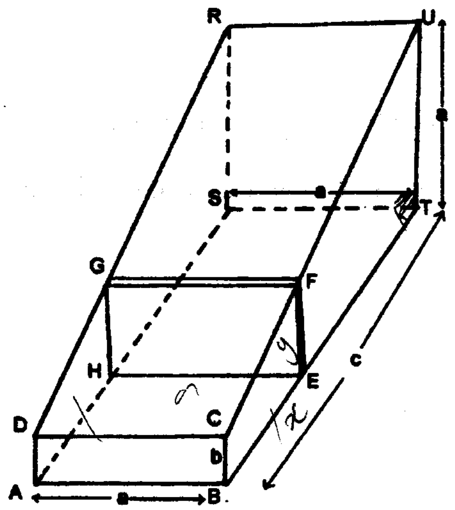
(i) $\{z : 1 \le |z| \le 3 \quad \text{and } 0 \le \arg z \le \frac{\pi}{2}\}$. **2**

(ii) $\{z : |z+1| + |z-1| = 3\}$. **3**

(iii) $\{z : \arg(z-2) - \arg(z+2) = \frac{\pi}{3}\}$. **3**

Question 4 (Start a new page)

(a) **5**

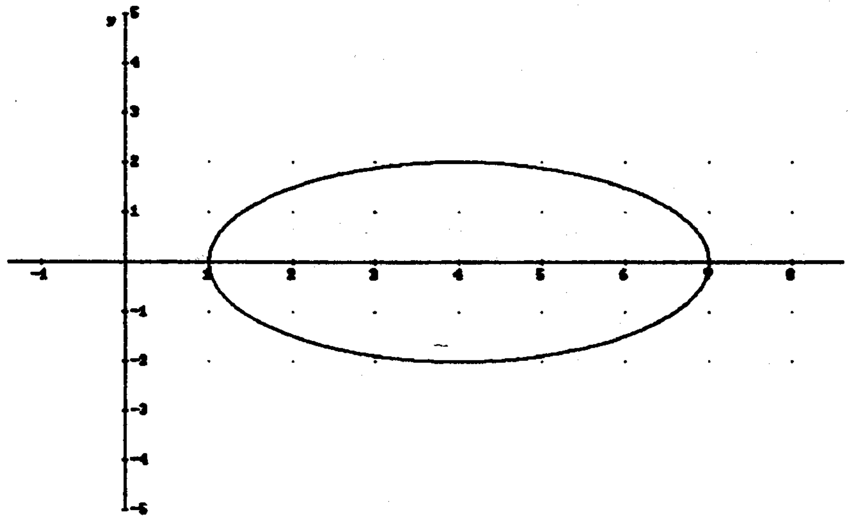


The diagram shows a solid with rectangular base ABTS. The end ABCD is a rectangle, and the other end STRU is a square. Both ends are perpendicular to the base.
Consider the slice of the solid with face HEFG and thickness Δx metres, and BE=AH= x metres.

(i) Show that the cross sectional area of this slice is $\frac{a}{c}[bc + (a-b)x]$.

(ii) Hence find the volume of the solid.

(b) The ellipse $\frac{(x-4)^2}{9}+\frac{y^2}{4}=1$ is rotated about the y axis forming a doughnut shape. **6**



(i) By taking slices perpendicular to the axis of rotation show that the volume of a slice is $8\pi\sqrt{36-9y^2}\,\delta y$.

(ii) Find the volume of the solid.

(c) Prove by induction that for every natural number n, if $A_1, A_2, \ldots\ldots\ldots A_n$ are pairwise distinct points, no three of which are on one line, then these points determine exactly $\frac{n(n-1)}{2}$ lines. **4**

Question 5 (Start a new page) **Marks**

(a) Let $f(x) = \sin x$, $-\pi \le x \le \pi$. Provide separate sketches of the graphs of:-

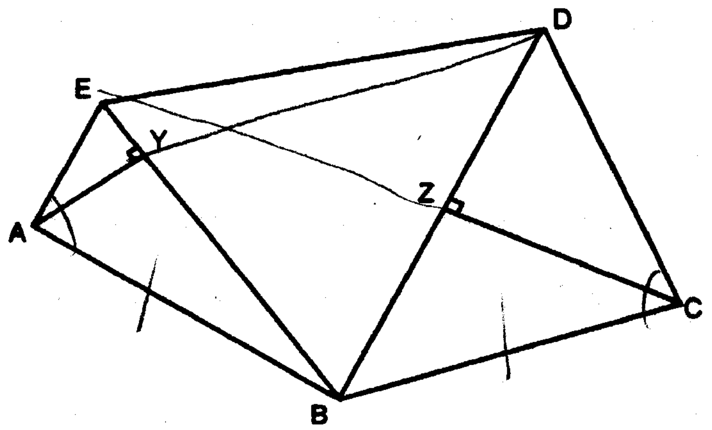
(i) $y = |f(x)|$. 2

(ii) $|y| = f(x)$. 2

(iii) $|y| = |f(x)|$. 2

(iv) $y = f(|x|)$. 2

(b) 7



ABCDE is a convex polygon such that AB=BC, $\angle BCD = \angle EAB = 90^\circ$. $AY \perp EB, BD \perp CZ$

(i) By using similar triangles prove that $AB^2 = BY.BE$.

(ii) Hence prove $\Delta BEZ \parallel\!| \Delta BDY$. (You may assume $BC^2 = BZ.BD$)

(iii) Show that $DEYZ$ is a cyclic quadrilateral.

Question 6 (Start a new page) **Marks**

(a) A boat is moving with constant speed in a circle of radius 60m. It does a complete circuit in 1 minute. The total mass of the boat is 300kg and the total resistance it meets is 600 Newtons.

(i) Show that θ, the angle made by the force (F) driving the boat and the tangent, is approximately 18°. 3

(ii) Hence find the force (F) driving the boat. 2

(b) A train of mass m, pulled by a locomotive which exerts a constant (propelling) force P is moving at speed v along a straight level track against a resistive force mkv, where k is a positive constant. Show that if the speed increases from 2m/s to 4m/s over a time interval of 5 seconds,

(i) $P = 2km(\frac{2e^{5k} - 1}{e^{5k} - 1})$. 4

(ii) Find the corresponding distance moved. 3

(iii) Prove that there is an upper bound to the speed that the train can attain and find the value of this upper bound. 3

Question 7 (Start a new page)

(a) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$ (chord PQ has positive slope)

(i) If the chord PQ subtends a right angle at the vertex of the parabola show that $pq = -4$. **2**

(ii) If the line PQ is inclined at an angle θ to the axis of the parabola, show that $\cot\theta = \dfrac{p+q}{2}$. **2**

(iii) Prove that the length of the chord is $4a\operatorname{cosec}\theta\sqrt{3+\operatorname{cosec}^2\theta}$ **4**

(b) Let $z = x + iy$ be a complex number (x and y real) satisfying $z\bar{z} + (1-2i)z + (1+2i)\bar{z} \le 3$

(i) Express the inequality in the Cartesian form. **2**

(ii) Sketch the locus of z on an Argand diagram. **1**

(iii) Find the maximum and minimum values of $x + y$ by considering lines of the form $x + y = k$. **4**

Question 8 (Start a new page)

(a) Prove $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. **2**

(b) In ΔABC, angle A is twice angle B, angle C is obtuse and the three lengths a, b, c are integers.

(i) Using the sine rule and the formulae for $\sin 2\beta$ and $\sin 3\beta$ show that $a^2 = b(b+c)$ **4**

(ii) Show $2\cos\beta = \dfrac{a}{b}$. **3**

(iii) If $\dfrac{n}{m} = \dfrac{a}{b}$ (where n and m have no common factor other than 1) deduce that $b = km^2$ and $b + c = kn^2$. **2**

(iv) Show that $\dfrac{\sqrt{3}}{2} < \cos\beta < 1$, and deduce that $m \ge 4$ and $n \ge 7$. **2**

(v) Find the minimum perimeter of ΔABC. **2**

END OF PAPER

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$