HURLSTONE AGRICULTURAL HIGH SCHOOL

PRO PATRIA

EXTENSION 2 MATHEMATICS

2013

YEAR 12

TRIAL (TASK 4) EXAMINATION

EXAMINERS ~ S. GEE, G. HUXLEY

GENERAL INSTRUCTIONS

- Reading Time – 5 minutes.
- Working Time – 3 hours.
- Attempt **all** questions.
- Each question in part B is worth 15 marks.
- **All** necessary working should be shown in every question in Part B.
- This paper contains Ten(10) multiple choice questions in Part A and Six(6) free response questions in Part B, totalling 14 pages including title page, multiple choice answer sheet and standard integral table.
- Board approved calculators and MathAids may be used.
- **Each free response question is to be started in a new answer booklet**. Write the question number and your student number at the top of each answer booklet.
- You **must** hand in the multiple choice answer sheet as well as an answer booklet for **each question** even if a question has not been attempted.
- This examination must **NOT** be removed from the examination room

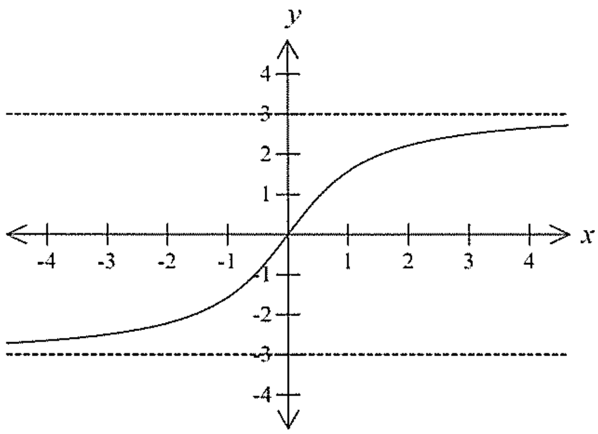
STUDENT NAME:______________________

TEACHER: ______________________

Part A   Multiple Choice (Complete on the answer sheet provided)

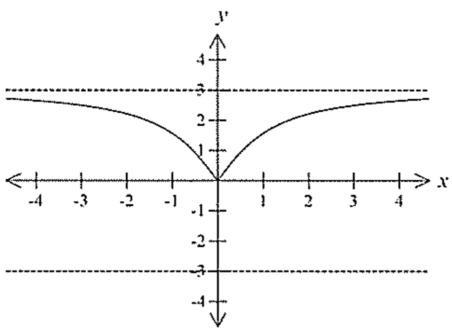
QUESTION 1

The diagram shows the graph of the function $y = f(x)$.

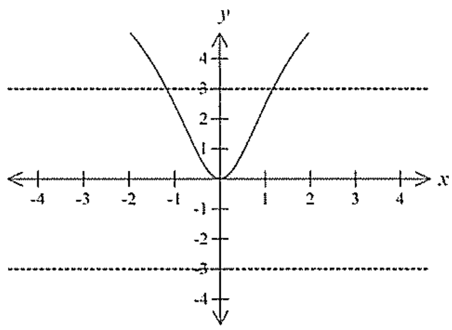


Which of the following is the graph of $y = \sqrt{f(x)}$?

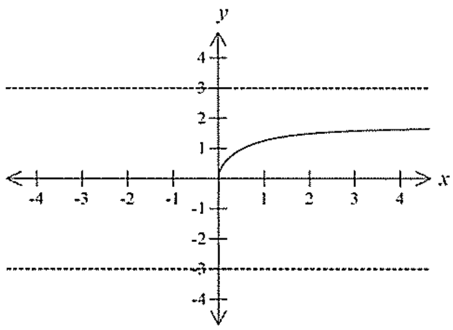
(A)



(B)



(C)



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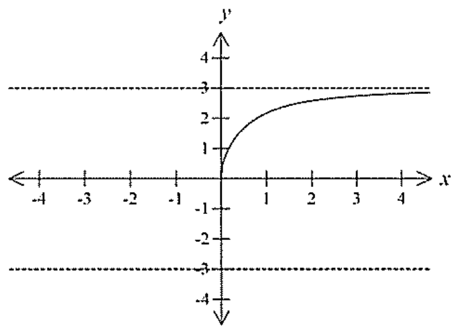
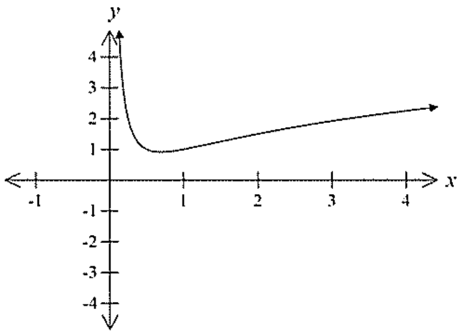


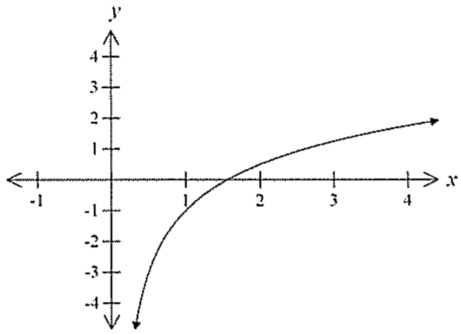
QUESTION 2

Which of the following is the sketch of $y = \log_2 x + \dfrac{1}{x}$?

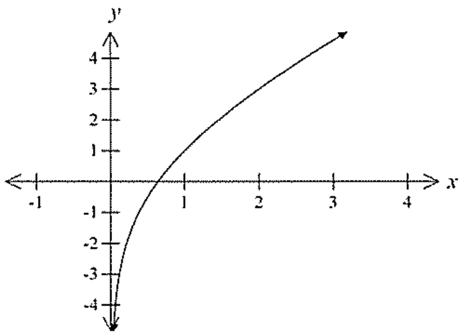
(A)



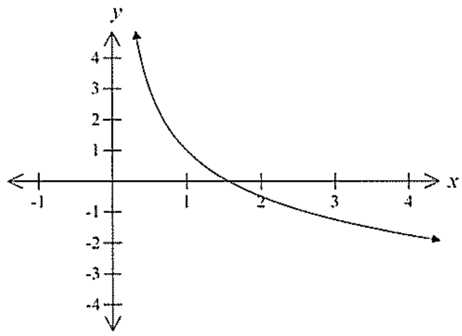
(B)



(C)

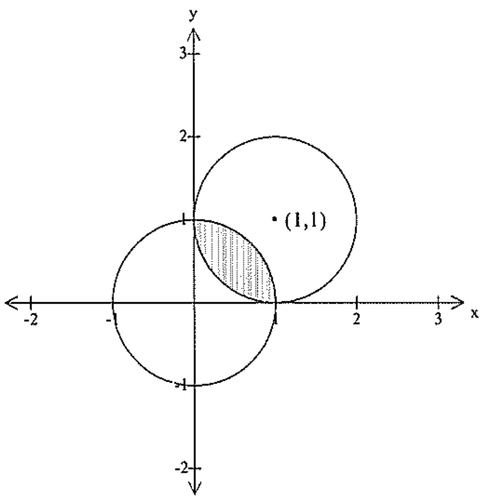


(D)



QUESTION 3

Consider the Argand diagram below.



Which inequality could define the shaded area?

(A) $|z| \leq 1$ and $|z-(1-i)| \geq 1$

(B) $|z| \leq 1$ and $|z-(1+i)| \geq 1$

(C) $|z| \leq 1$ and $|z-(1-i)| \leq 1$

(D) $|z| \leq 1$ and $|z-(1+i)| \leq 1$

QUESTION 4

It is given that $3+i$ is a root of $P(z)=z^3+az^2+bz+10$ where a and b are real numbers. Which expression factorises $P(z)$ over the real numbers?

(A) $(z-1)(z^2+6z-10)$

(B) $(z-1)(z^2-6z-10)$

(C) $(z+1)(z^2+6z+10)$

(D) $(z+1)(z^2-6z+10)$

QUESTION 5

How many distinct permutations of the letters of the word 'ATTAINS' are possible in a straight line when the word begins and ends with the letter T?

(A) 60

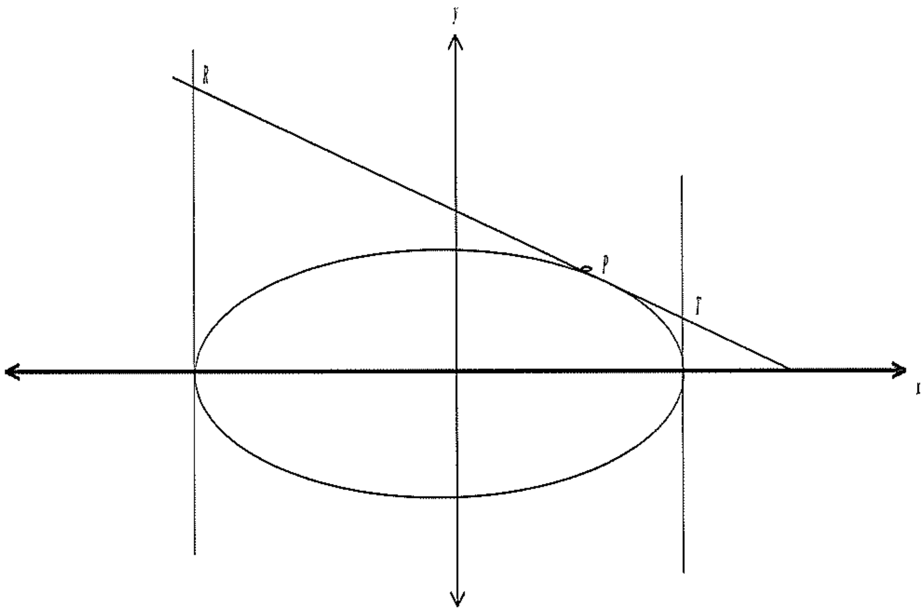
(B) 120

(C) 360

(D) 1260

QUESTION 6

The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $a>b>0$.



What is the equation of the tangent at P?

(A) $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$

(B) $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$

(C) $\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$

(D) $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$

QUESTION 7

What is the value of $\int_0^1 \frac{e^x}{1+e^x}\,dx$?

(A) $\log_e(1+e)$

(B) 1

(C) $\log_e\left(\frac{1+e}{2}\right)$

(D) $\log_e\frac{e}{2}-2$

QUESTION 8

The equation $24x^3 - 12x^2 - 6x + 1$ has roots α, β and γ.
What is the value of α if $\alpha = \beta + \gamma$?

(A) $-\frac{1}{2}$

(B) $\frac{1}{4}$

(C) $\frac{1}{2}$

(D) 1

QUESTION 9

Consider the hyperbola with the equation $\frac{x^2}{144} - \frac{y^2}{25} = 1$.
What are the equations of the directrices?

(A) $x = \pm\frac{13}{144}$

(B) $x = \pm\frac{13}{25}$

(C) $x = \pm\frac{25}{13}$

(D) $x = \pm\frac{144}{13}$

QUESTION 10

The polynomial $P(x) = x^4 + ax^2 + bx + 28$ has a double root at $x = 2$.
What are the values of a and b?

(A) $a = -11$ and $b = -12$
(B) $a = -5$ and $b = -12$
(C) $a = -11$ and $b = 12$
(D) $a = -5$ and $b = 12$

QUESTION 11 (Commence a new answer booklet) **Marks**

(a) For the complex number $w = 1 - i\sqrt{3}$:

(i) Find $|w|$ and $\arg(w)$ **2**

(ii) Express $\overline{w}, w^2, \dfrac{1}{w}$ and $\sqrt{w}$ in the form $a + ib$.
Plot them on the Argand diagram. **5**

(b) Describe and sketch the locus of the point z such that $|z + 3i| + |z - 3i| = 10$ **2**

(c) (i) Show that $(1 - 2i)^2 = -3 - 4i$. **1**

(ii) Hence solve the equation $x^2 - 5x + (7 + i) = 0$ **2**

(d) Let $OABC$ be a square on the Argand diagram where O is the origin.
The points A and C represent the complex numbers z and iz respectively.

(i) Find the complex number represented by B. **1**

(ii) The square is now rotated through $45°$ in an anticlockwise direction about O to $OA'B'C'$. Find the complex number represented by A'. **2**

QUESTION 12 (Commence a new answer booklet) **Marks**

(a) For the curve, $f(x) = \frac{4x}{1+x^2}$,

(i) Prove the curve $y = f(x)$ has a relative minimum at A (–1,–2), a relative maximum at B (1,2) and a point of inflexion at O (0,0). Sketch $y = f(x)$ **3**

(ii) The cubic curve $g(x) = ax^3 + bx^2 + cx + d$ also has a relative minimum at A (–1,–2) and a relative maximum at B (1,2).

(α) Obtain values of the coefficients a, b, c, and d. **2**

(β) Deduce that O is also a point of inflexion on $y = g(x)$ **1**

(iii) Prove that the two curves $y = f(x)$ and $y = g(x)$ have only the three points A, B and O in common. **2**

(b) Given $h(x) = 6 + x - x^2$
Draw neat sketches, on separate diagrams, of at least one third of a page for

(i) $y = h(x)$ **1**

(ii) $y = |h(x)|$ **1**

(iii) $y = \sqrt{h(x)}$ **1**

(iv) $y^2 = h(x)$ **1**

(v) $y = \frac{1}{h(x)}$ **1**

(vi) $y = e^{h(x)}$ **1**

(vii) $y = \log_2 h(x)$ **1**

QUESTION 13 (Commence a new answer booklet) **Marks**

(a) Consider the ellipse E whose equation is $\frac{x^2}{4}+\frac{y^2}{2}=1$.

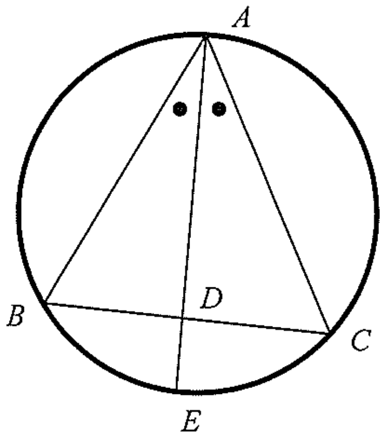
(i) Show that the equation of E may be written in the parametric form

$$x = 2\cos\theta$$
$$y = \sqrt{2}\sin\theta$$
1

(ii) Assuming that the perimeter of E is given by the formula $p = 2\int_0^{\pi}\left[\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\right]d\theta$ show that $p = 2\sqrt{2}\int_0^{\pi}\left[\sqrt{2-\cos^2\theta}\right]d\theta$ **2**

(iii) Use five evenly spaced function values from $\theta = 0$ to $\theta = \pi$ and Simpson's rule to estimate p correct to two decimal places. **2**

(b)



In the diagram, the bisector AD of $\angle BAC$ has been extended to intersect the circle ABC at E.

(i) Prove that the triangles ABE and ADC are similar. **2**

(ii) Show that $AB.AC=AD.AE$. **1**

(iii) Prove that $AD^2 = AB.AC - BD.DC$. **2**

(c) Given $a_n = \sqrt{2+a_{n-1}}$ for integers $n \geq 1$, and that $a_0 = 1$, use the process of mathematical induction to prove for $n \geq 1, \sqrt{2} < a_n < 2$. **3**

(d) A woman travelling along a straight flat road passes three points at intervals of 200 metres. From these points she observes the angle of elevation of the top of the hill to the left of the road to be respectively $30°, 45°$ and again $45°$. Find the height of the hill to the nearest metre. **2**

QUESTION 14 (Commence a new answer booklet) **Marks**

(a) $P\left(pt, \frac{t}{p}\right)$ and $Q\left(qt, \frac{t}{q}\right)$ are two points on the rectangular hyperbola: $xy = t^2$, where p and q are constants.

(i) Show that the gradient of PQ is $\frac{-1}{pq}$ **1**

(ii) Show that the gradient of the tangent to the hyperbola at P is $\frac{-1}{p^2}$ **1**

(iii) Hence, or otherwise, determine an expression for q in terms of p that will Make PQ a normal to the hyperbola at P. **2**

(b) For the hyperbola $5x^2 - 4y^2 = 20$

(i) Find the eccentricity and the co-ordinates of the foci. **2**

(ii) Find the equations of the asymptotes. **1**

(c) When the polynomial $P(x)$ is divided by $(x+2)(x-3)$ the remainder is $4x+1$.

What is the remainder when $P(x)$ is divided by $(x+2)$? **2**

(d) (i) Show that $2+i$ is a zero of $x^3 - 11x + 20 = 0$. **2**

(ii) Hence, or otherwise, solve $x^3 - 11x + 20 = 0$. **2**

(e) The equation $x^3 + 2x - 1 = 0$ has roots α, β, δ.

Find the polynomial equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\delta}$. **2**

QUESTION 15 (Commence a new answer booklet) **Marks**

(a) Evaluate the integral $\int_{-2}^{2} x\sqrt{4-x^2} - \sqrt{4-x^2}\,dx$ **2**

(b) Using integration by parts, evaluate: $\int_{0}^{1} x\tan^{-1}x\,dx$ **3**

(c) (i) Find the real numbers a, b, c such that $\dfrac{5}{x^2(2-x)} \equiv \dfrac{ax+b}{x^2} + \dfrac{c}{2-x}$ **2**

(ii) Hence, or otherwise, find $\int \dfrac{20}{x^2(2-x)}\,dx$ **3**

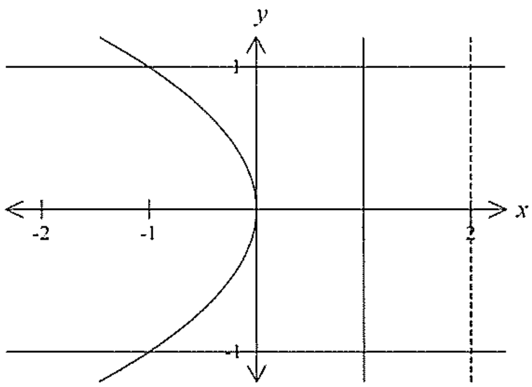
(d) Let $I_n = \int_{1}^{2} (\ln x)^n\,dx$, where n is a positive integer.

(i) Show that $I_n = 2(\ln 2)^n - nI_{n-1}$ **2**

(ii) Hence evaluate $I_4 = \int_{1}^{2} (\ln x)^4\,dx$. Write your answer in exact form. **3**

QUESTION 16 (Commence a new answer booklet) **Marks**

(a) (i) The region bounded by the lines $x=1$, $y=1$, $y=-1$ and by the curve $x=-y^2$ is rotated through 360° about the line $x=2$ to form a solid.



By considering summation of slices in the shape of circular discs, show that the correct expression for the volume of the solid generated is:

$$V=\int_{-1}^{1}\pi\left(y^4+4y^2+4\right)dy$$ **3**

(ii) Hence or otherwise, evaluate the volume of the solid of revolution. **2**

(b) (i) Draw a neat sketch of the region enclosed by the curve $y=4x-x^2$, the x – axis, and the lines $x=1$ and $x=3$. Include in your diagram all points of intersection. **2**

(ii) Use the method of cylindrical shells to find the volume of the solid generated when the area in part (i) is rotated about the y-axis. **3**

(c) (i) On a number plane, draw the region bounded by: the curve $y^2=4x$ and the lines $x+y=0$ and $x=4$. **1**

(ii) A solid is generated using the region in (i) as a base.
There are square cross-sectional slices perpendicular to the x – axis.
Each has a side with one end-point on the line $x+y=0$ and the other on the curve $y^2=4x$.

(α) Show that the area of a cross-section is given by:

$$A(x)=4x+x^2+4x^{\frac{3}{2}}.$$ **2**

(β) Hence find the volume of the solid formed. **2**

END OF EXAMINATION

STUDENT NUMBER:…………………….. ……………..

HURLSTONE AGRICULTURAL HIGH SCHOOL

2013 TRIAL EXAMINATION
YEAR 12
EXTENSION 2
MATHEMATICS

PART A Answers

1	ⓐ	ⓑ	ⓒ	ⓓ
2	ⓐ	ⓑ	ⓒ	ⓓ
3	ⓐ	ⓑ	ⓒ	ⓓ
4	ⓐ	ⓑ	ⓒ	ⓓ
5	ⓐ	ⓑ	ⓒ	ⓓ
6	ⓐ	ⓑ	ⓒ	ⓓ
7	ⓐ	ⓑ	ⓒ	ⓓ
8	ⓐ	ⓑ	ⓒ	ⓓ
9	ⓐ	ⓑ	ⓒ	ⓓ
10	ⓐ	ⓑ	ⓒ	ⓓ

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, \quad n \neq -1; \; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x + C, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax + C, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax + C, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax + C, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax + C, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a} + C, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right) + C, \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

NOTE : $\ln x = \log_e x, \quad x > 0$