HURLSTONE AGRICULTURAL HIGH SCHOOL

YEAR 12

MATHEMATICS EXTENSION 2

2005

HSC COURSE

TRIAL HSC ~ASSESSMENT TASK 4

EXAMINERS ~ J. DILLON AND G. RAWSON

GENERAL INSTRUCTIONS

- READING TIME – 5 MINUTES.
- WORKING TIME – THREE HOURS.
- ATTEMPT ALL QUESTIONS.
- QUESTIONS ARE OF EQUAL VALUE.
- ALL NECESSARY WORKING SHOULD BE SHOWN IN EVERY QUESTION.
- THIS PAPER CONTAINS EIGHT (8) QUESTIONS.
- TOTAL MARKS – 120 MARKS
- MARKS MAY NOT BE AWARDED FOR CARELESS OR BADLY ARRANGED WORK.
- BOARD APPROVED CALCULATORS MAY BE USED.
- A TABLE OF INTEGRALS IS SUPPLIED.
- EACH QUESTION IS TO BE STARTED IN A NEW EXAMINATION BOOKLET.
- THIS ASSESSMENT TASK MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

STUDENT NAME / NUMBER: ____________________

TEACHER: ____________________

QUESTION 1: (USE A SEPARATE ANSWER BOOKLET)

MARKS

(a) Find:

(i) $\displaystyle\int \frac{1}{x \ln x}\, dx$ **2**

(ii) $\displaystyle\int \frac{x}{x^2 + 2x + 5}\, dx$ **3**

(b) Evaluate, using integration by parts, $\displaystyle\int_0^{\frac{\pi}{2}} x \cos x \, dx$. **2**

(c) Evaluate, using partial fractions, $\displaystyle\int_1^3 \frac{dx}{x^2 + 2x}$. **3**

(d) The integral I_n is defined by $I_n = \displaystyle\int_0^1 x^n e^{-x}\, dx$.

(i) Show that $I_n = nI_{n-1} - e^{-1}$. **2**

(ii) Hence, show that $I_3 = 6 - 16e^{-1}$. **3**

Question 2: (Use a SEPARATE answer booklet)

Marks

(a) Given $z = \sqrt{6} - \sqrt{2}\, i$, find:

(i) $\operatorname{Re}(z^2)$; **1**

(ii) $|z|$; **1**

(iii) $\arg z$; **2**

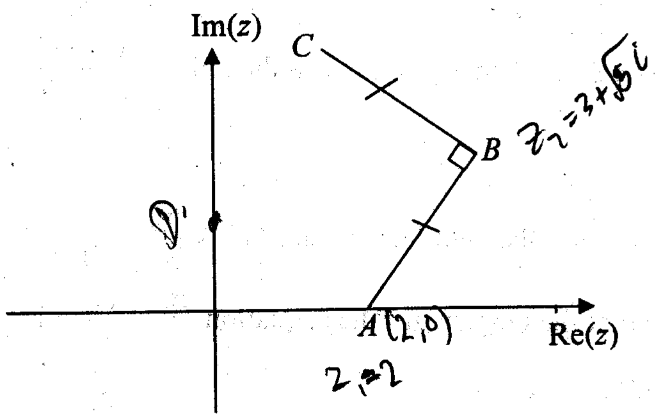
(iv) z^4 in the form $x + iy$. **2**

(b) The equations $|z - 8 - 6i| = 2\sqrt{10}$ and $\arg z = \tan^{-1} 2$ both represent loci on the Argand plane.

(i) Write down the Cartesian equations of the loci, and hence show that the points of intersection of the loci are $2 + 4i$ and $6 + 12i$. **3**

(ii) Sketch both loci on the same diagram, showing their points of intersection. (You need not show the intercepts with the axes.) **2**

(c)



The diagram above shows the fixed points A, B and C in the Argand plane, where $AB = BC$, $\angle ABC = \frac{\pi}{2}$, and A, B and C are in anticlockwise order. The point A represents the complex number $z_1 = 2$ and the point B represents the complex number $z_2 = 3 + \sqrt{5}\, i$.

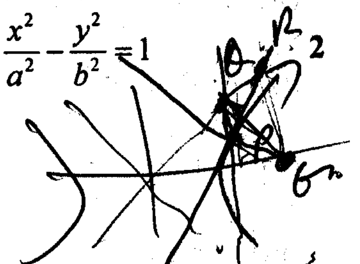
(i) Find the complex number z_3 represented by the point C. **2**

(ii) D is the point on the Argand plane such that $ABCD$ is a square. Find the complex number z_4 represented by D. **2**

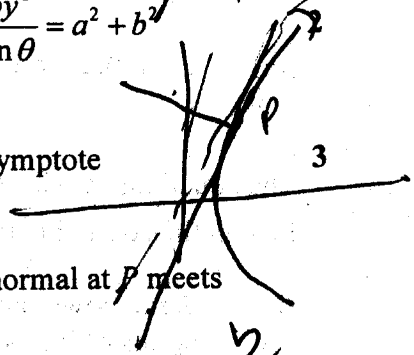
QUESTION 3: (USE A SEPARATE ANSWER BOOKLET)

MARKS

(a) (i) Show that the equation of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $P(a\sec\theta, b\tan\theta)$ is $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$. **2**



(ii) Show that the equation of the normal at P is $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$. **2**



(iii) The line through P parallel to the y – axis meets the asymptote $y=\frac{b}{a}x$ at Q.
The tangent at P meets the same asymptote at R. The normal at P meets the x – axis at G. Prove that $\angle RQG$ is a right angle. **3**

(iv) What sort of quadrilateral is $RQPG$? **1**

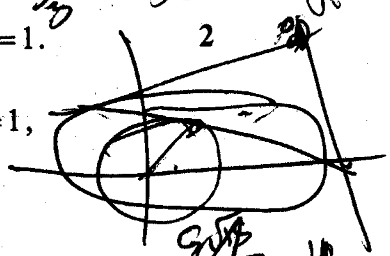
$m_{PQ} = \frac{b}{a}$

$b^2 = a^2(1-e^2)$

(b) The tangents at two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ intersect at $T(x_0, y_0)$.

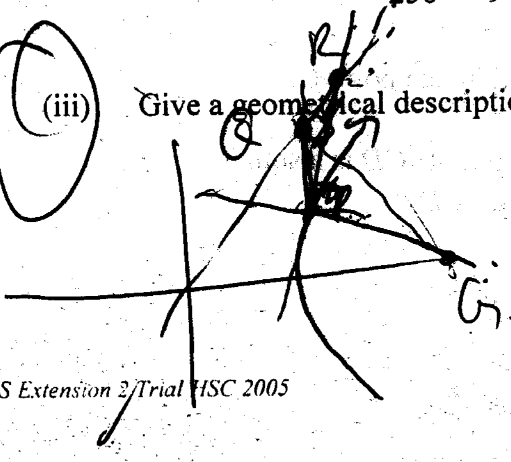
(i) Show that the equation of the chord of contact PQ is $\frac{xx_0}{16}+\frac{yy_0}{9}=1$. **2**

(You may assume that the tangent at P has equation $\frac{xx_1}{16}+\frac{yy_1}{9}=1$, and similarly for Q)



(ii) If the chord PQ touches the circle $x^2+y^2=9$, then by considering the distance of the chord from the origin, or otherwise, show that the point $T(x_0, y_0)$ satisfies $\frac{9x_0^2}{256}+\frac{y_0^2}{9}=1$. **4**

(iii) Give a geometrical description of the locus of T. **1**



$e^2 = \frac{175}{256}$

QUESTION 4: (USE A SEPARATE ANSWER BOOKLET)

MARKS

(a) (i) Let $P(x)$ be a polynomial of degree 4 with a zero of multiplicity 3. Show that $P'(x)$ has a zero of multiplicity 2. **2**

(ii) Hence, or otherwise, find all zeros of $P(x) = 8x^4 - 25x^3 + 27x^2 - 11x + 1$, given that it has a zero of multiplicity 3. **2**

$1, \frac{1}{8}$

(iii) Sketch $y = 8x^4 - 25x^3 + 27x^2 - 11x + 1$, clearly showing the intercepts on the coordinate axes. ***You do not need to give the coordinates of turning points or points of inflection.*** **1**

$= \text{cis}(\pi(2k+1))$

5θ

(b) (i) Show that the general solution of the equation $\cos 5\theta = -1$ is given by **2**

$$\theta = (2n+1)\frac{\pi}{5},\ n = 0, \pm 1, \pm 2, \ldots .$$

Hence, solve the equation $\cos 5\theta = -1$, for $0 \le \theta \le 2\pi$.

(ii) Use De Moirve's Theorem to show that **3**

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$

(iii) Find the exact trigonometric roots of the equation **2**

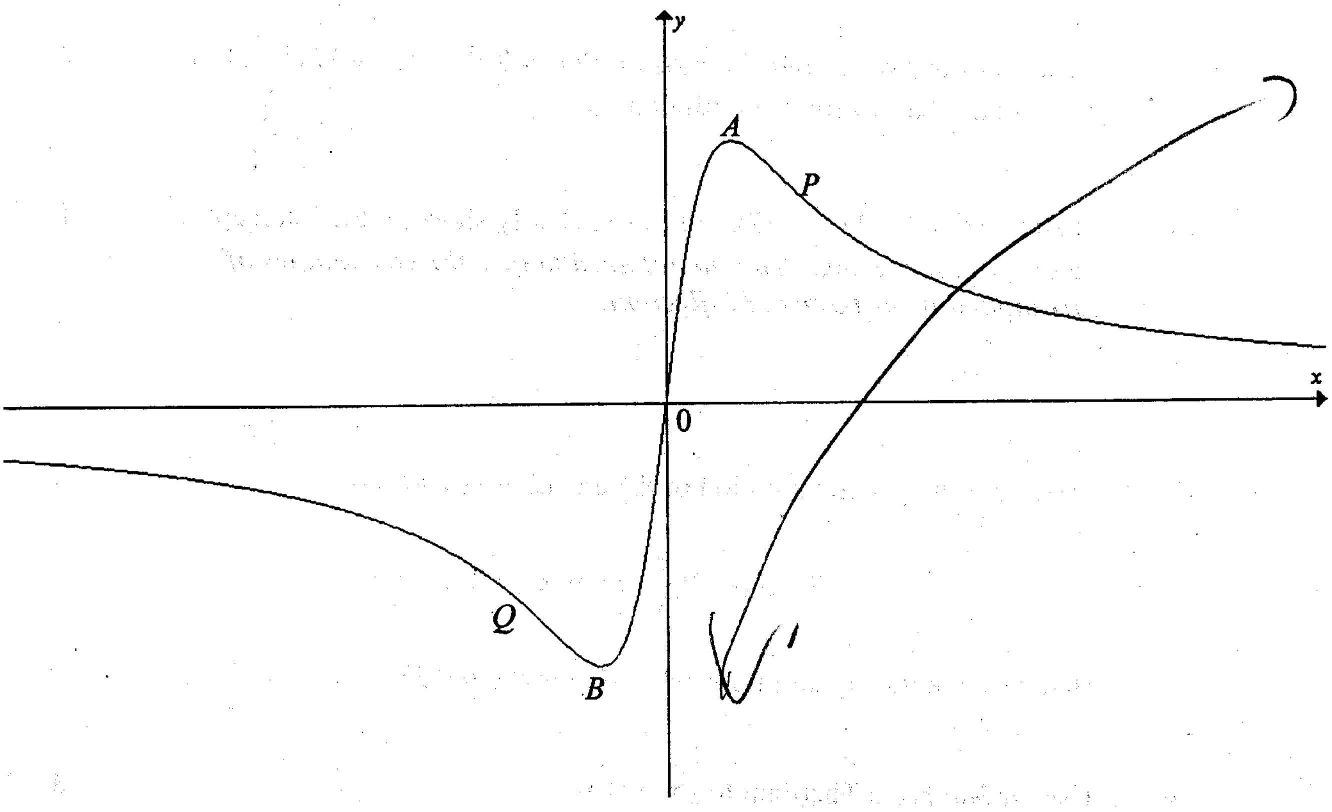
$$16x^5 - 20x^3 + 5x + 1 = 0$$

(iv) Hence, find the exact values of $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}$ and $\cos\frac{\pi}{5} \times \cos\frac{3\pi}{5}$ and factorise $16x^5 - 20x^3 + 5x + 1 = 0$ into irreducible factors over the rational numbers. **3**

QUESTION 5: (USE A SEPARATE ANSWER BOOKLET)

MARKS

(a) In the diagram below, the curve $y=\dfrac{2x}{1+x^2}$ is sketched.



(i) Find the coordinates of the turning points A and B. **2**

(ii) Find the coordinates of the inflection points ~~X~~ P and ~~R~~ Q. **3**

(b) Draw separate sketches of:

(i) $y=\dfrac{|2x|}{1+x^2}$ **1**

(ii) $y=\dfrac{1+x^2}{2x}$ $= \dfrac{x}{2} + \dfrac{1}{2x}$ **2**

(iii) $y^2=\dfrac{2x}{1+x^2}$ **2**

(iv) $y=\log_e\left(\dfrac{2x}{1+x^2}\right)$ **2**

QUESTION 5 CONTINUES ON THE NEXT PAGE ....

QUESTION 4: (USE A SEPARATE ANSWER BOOKLET)

MARKS

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(ii) Hence, or otherwise, find all zeros of $P(x) = 8x^4 - 25x^3 + 27x^2 - 11x + 1$, given that it has a zero of multiplicity 3. **2**

$1, \frac{1}{8}$

(iii) Sketch $y = 8x^4 - 25x^3 + 27x^2 - 11x + 1$, clearly showing the intercepts on the coordinate axes. ***You do not need to give the coordinates of turning points or points of inflection.*** **1**

$= \text{cis}[\pi(2k+1)]$

$5\theta = \ldots$

(b) (i) Show that the general solution of the equation $\cos 5\theta = -1$ is given by **2**

$$\theta = (2n+1)\frac{\pi}{5}, n = 0, \pm 1, \pm 2, \ldots .$$

Hence, solve the equation $\cos 5\theta = -1$, for $0 \le \theta \le 2\pi$.

(ii) Use De Moirve's Theorem to show that **3**

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta .$$

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(iv) Hence, find the exact values of $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}$ and $\cos\frac{\pi}{5} \times \cos\frac{3\pi}{5}$ and factorise $16x^5 - 20x^3 + 5x + 1 = 0$ into irreducible factors over the rational numbers. **3**

(c) (i) Show that the equation $kx^3+(k-2)x=0$ can be written in the form $\dfrac{2x}{1+x^2}=kx$. **1**

(ii) Using a graphical approach based on the curve $y=\dfrac{2x}{1+x^2}$, or otherwise, find the real values of k for which the equation $kx^3+(k-2)x=0$ has exactly one real root. **2**

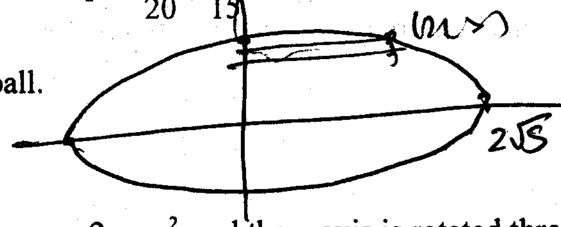
QUESTION 6: (USE A SEPARATE ANSWER BOOKLET)

$x^2 = 20\left(1-\frac{y^2}{15}\right)$

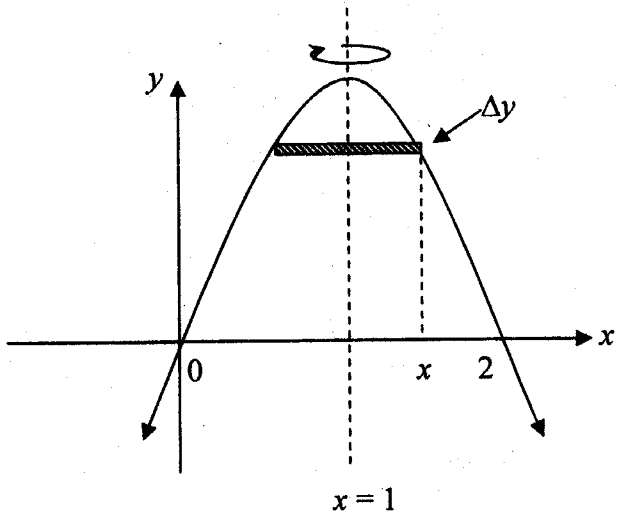
MARKS

(a) A Mini-League football has a volume the same as the volume generated by rotating the region inside the ellipse $\frac{x^2}{20}+\frac{y^2}{15}=1$ about the x-axis. **3**

Find the volume of this football.



(b) The area bounded by the curve $y = 2x - x^2$ and the x-axis is rotated through 180° about the line $x = 1$.



(i) Show that the volume, ΔV, of a representative horizontal slice of width Δy is given by **2**

$$\Delta V = \pi(x-1)^2 \Delta y$$

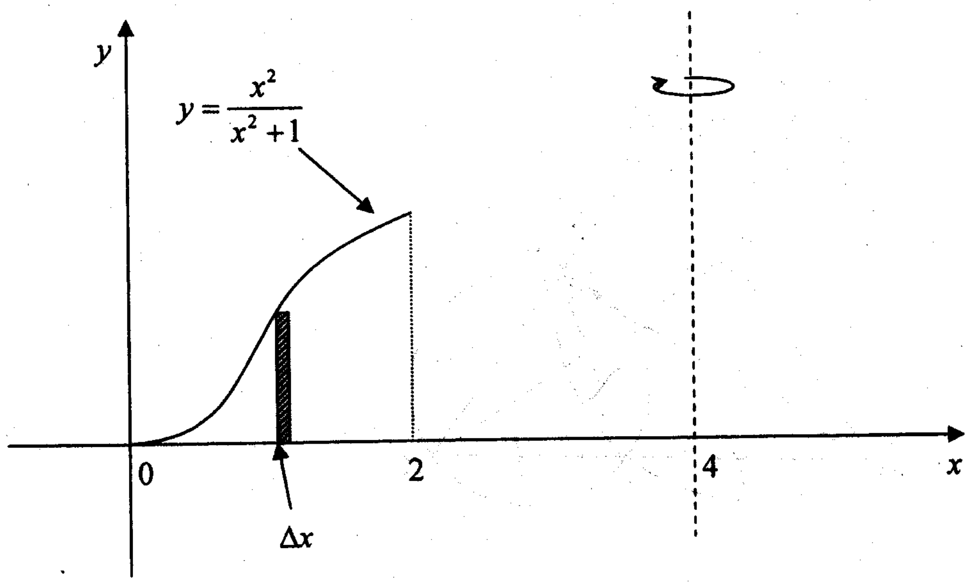
(ii) Hence show that the volume of the solid of revolution is given by **2**

$$V = \lim_{\Delta y \to 0} \sum_{y=0}^{1} \pi(1-y)\Delta y$$

(iii) Hence, find the volume of the solid of revolution **2**

QUESTION 6 CONTINUES ON THE NEXT PAGE ....

(c) The region shown below, bounded by the curve $y = \dfrac{x^2}{x^2+1}$, the x-axis and the line $x = 2$, is rotated about the line $x = 4$.



(i) Using the method of cylindrical shells, show that the volume ΔV of a Shell distant x from the origin and with thickness Δx is given by **3**

$$\Delta V = 2\pi(4-x)(1-\frac{1}{1+x^2})\Delta x$$

(ii) Hence, find the volume of the solid **3**

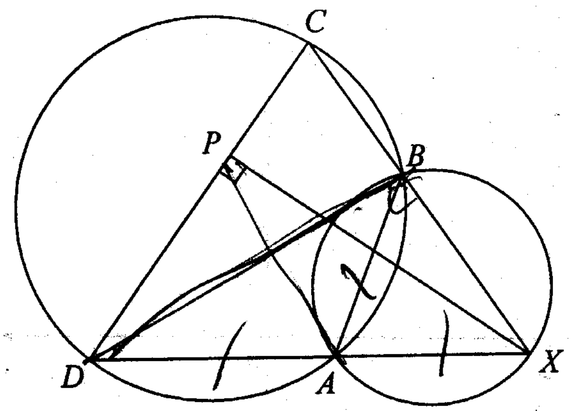
$$V = 2\pi \int (4-x)\left(1 - \frac{1}{1+x^2}\right) dx$$

$$= 2\pi \int \left(4 - \frac{4}{1+x^2} - x + \frac{1}{1+x^2}\right) dx$$

QUESTION 7: (USE A SEPARATE ANSWER BOOKLET)

MARKS

(a)



In the diagram above, $AB = AD = AX$ and $XP \perp DC$.

(i) Prove that $\angle DBX = 90°$ **2**

(ii) Hence, or otherwise, prove that $AB = AP$. **3**

(b) (i) Show that $a^2 + b^2 > 2ab$, where a and b are distinct positive real numbers. **1**

(ii) Hence show that $a^2 + b^2 + c^2 > ab + bc + ca$, where a, b and c are distinct positive real numbers. **2**

(iii) Hence, or otherwise, prove that **2**

$$\frac{a^2b^2 + b^2c^2 + c^2a^2}{a+b+c} > abc,$$

where a, b and c are distinct positive real numbers.

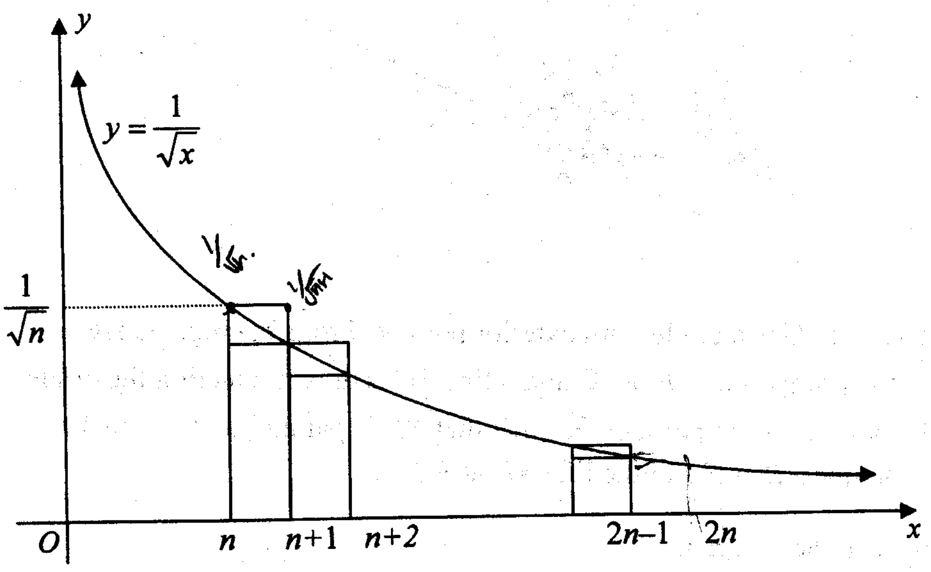
(c) A sequence, T_n, is such that $T_1 = 3$, $T_2 = 5$ and $T_{n+2} = 4T_{n+1} - 3T_n$. **5**
Prove by mathematical induction that $T_n = 3^{n-1} + 2$.

QUESTION 8: (USE A SEPARATE ANSWER BOOKLET)

MARKS

(a) (i) Show that $\displaystyle\int_n^{2n} \frac{dx}{\sqrt{x}} = 2\sqrt{n}\left(\sqrt{2}-1\right)$. **2**

(ii)



In the diagram above, the graph of $y = \dfrac{1}{\sqrt{x}}$ has been drawn, and n upper and lower rectangles have been constructed between $x = n$ and $x = 2n$, each of width 1 unit.

Let $S_n = \dfrac{1}{\sqrt{n+1}} + \dfrac{1}{\sqrt{n+2}} + \dfrac{1}{\sqrt{n+3}} + \cdots + \dfrac{1}{\sqrt{2n}}$.

(α) By considering the sums of areas of upper and lower rectangles, show that: **4**

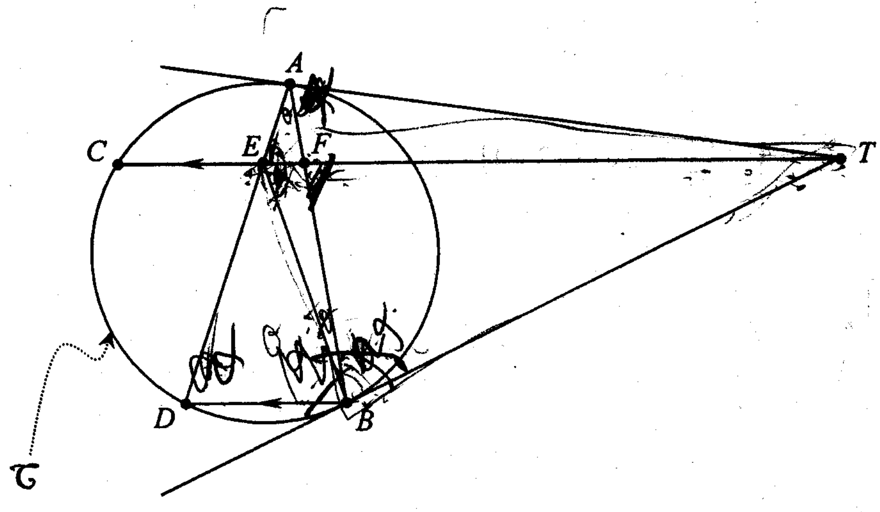
$$2\sqrt{n}\left(\sqrt{2}-1\right) + \frac{1-\sqrt{2}}{\sqrt{2n}} < S_n < 2\sqrt{n}\left(\sqrt{2}-1\right)$$

(β) Hence find, correct to four decimal places, **1**

$$\frac{1}{\sqrt{10^8+1}} + \frac{1}{\sqrt{10^8+2}} + \frac{1}{\sqrt{10^8+3}} + \cdots + \frac{1}{\sqrt{2\times 10^8}}$$

QUESTION 8 CONTINUES ON THE NEXT PAGE ....

(b)



In the diagram, 𝒞 is a circle with exterior point T. From T, tangents are drawn to the points A and B on 𝒞 and a line TC is drawn, meeting the circle at C. The point D is the point on 𝒞 such that BD is parallel to TC. The line TC cuts the line AB at F and the line AD at E.

Copy or trace the diagram.

(i) Prove that ΔTFA is similar to ΔTAE. **3**

(ii) Deduce that $TE.TF = TB^2$. **2**

(iii) Show that ΔEBT is similar to ΔBFT. **2**

(iv) Prove that ΔDEB is isosceles. **1**

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, \quad n \neq -1; \; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x + C, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + C, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax + C, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax + C, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax + C, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax + C, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a} + C, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right) + C, \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

NOTE : $\ln x = \log_e x, \quad x > 0$