HORNSBY GIRLS HIGH SCHOOL

Mathematics Extension 2

Year 12 Higher School Certificate
Trial Examination Term 3 2013

STUDENT NUMBER: ______________________

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations
- Marks may be deducted for untidy and poorly arranged work
- Do not use correction fluid or tape
- Do not remove this paper from the examination

Total marks – 100

Section I Pages 3 – 6

10 marks

Attempt Questions 1 – 10

Answer on the Objective Response Answer Sheet provided

Section II Pages 7 – 15

90 marks

Attempt Questions 11 – 16.

Start each question in a new writing booklet.

Write your student number on every writing booklet.

Question	***1-10***	***11***	***12***	***13***	***14***	***15***	***16***	***Total***
Total	*/10*	*/15*	*/15*	*/15*	*/15*	*/15*	*/15*	*/100*

This assessment task constitutes 45% of the Higher School Certificate Course School Assessment

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Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the Objective Response answer sheet for Questions 1 – 10

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1 Let $z = 3 + 2i$ and $w = 2 - 3i$. What is the value of $3\overline{z} - 2w$?

(A) 5

(B) -5

(C) $5 + 12i$

(D) $5 - 12i$

2 The equation $x^2 + 2y^2 - 2xy + x = 8$ defines y implicitly as a function of x.

What is the value of $\frac{dy}{dx}$ at the point $(3,2)$?

(A) $\frac{1}{4}$

(B) $-\frac{1}{4}$

(C) $\frac{3}{2}$

(D) $-\frac{3}{2}$

3 Let $z = \cos\theta + i\sin\theta$. Which of the following is equal to z^3?

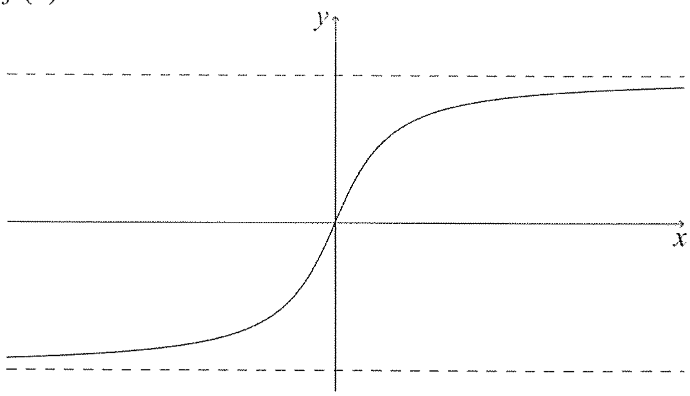
(A) $\cos^3\theta + i\sin^3\theta$

(B) $\cos^3\theta - i\sin^3\theta$

(C) $\cos 3\theta + i\sin 3\theta$

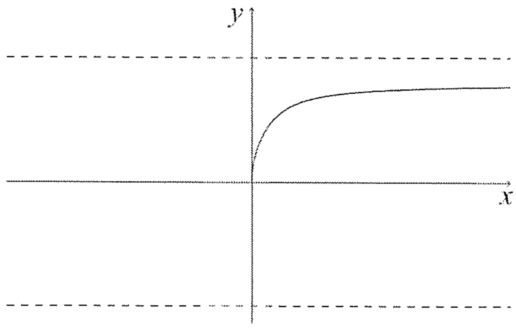
(D) $\cos 3\theta - i\sin 3\theta$

4 The graph of $y = f(x)$ is shown below.



Which of the following graphs best represents $y^2 = f(x)$?

(A)



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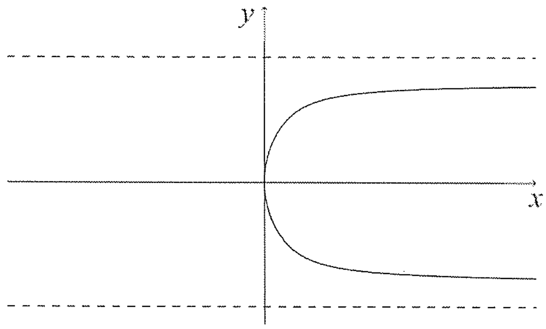
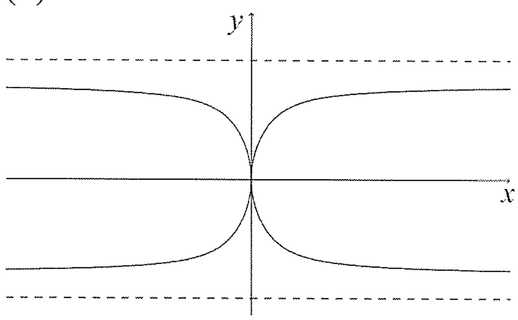
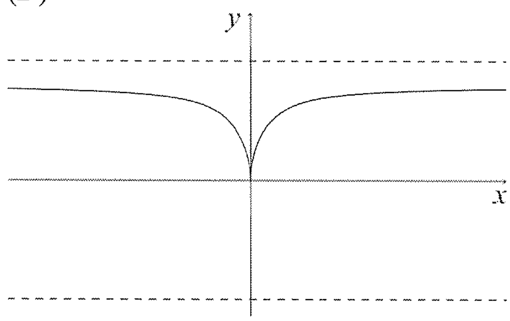


(C)



(D)



5 The roots of the polynomial $4x^3 + 4x - 5 = 0$ are α, β and γ.
What is the value of $(\alpha+\beta-3\gamma)(\beta+\gamma-3\alpha)(\alpha+\gamma-3\beta)$?

(A) -80

(B) -16

(C) 16

(D) 80

6 A mass of 5 kg moves in a horizontal circle of radius 1.5 metres at a uniform angular speed of 4 radians per second. What is the centripetal force required for this motion?

(A) 40N

(B) 80N

(C) 120N

(D) 160N

7 Which of the following is a focus of the hyperbola $\frac{x^2}{11}-\frac{y^2}{25}=-1$?

(A) $(0,5)$

(B) $(5,0)$

(C) $(6,0)$

(D) $(0,6)$

8 If $x^3-11x^2+40x-k=(x-4)^2.P(x)$, what is the value of k ?

(A) 16

(B) 32

(C) 48

(D) 64

9 The region bounded by the curve $y=x^2$, the line $x=4$ and the x-axis is rotated about the line $x=4$. Which integral represents the volume of the solid?

(A) $2\pi\int_0^4 (4-x)x^2 dx$

(B) $\pi\int_0^{16} (4-x)x^2 dx$

(C) $2\pi\int_0^4 (4-x)^2 dx$

(D) $\pi\int_0^{16} (4-x)^2 dx$

10 Without evaluating the integrals, which of the following integrals is equal to zero?

(A) $\int_{-1}^{1} e^{-x} \tan^{-1}(x^2)dx$

(B) $\int_{-1}^{1} \frac{x^2 \sin x}{x^2+5} dx$

(C) $\int_{-1}^{1} \sqrt{x^2+e^x}\ dx$

(D) $\int_{-1}^{1} x^3 \sin^{-1} x\ dx$

End of Section I

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section

Answer each question in a new writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations

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Question 11 (15 marks) Start a new writing booklet

(a) (i) Using the substitution $x = a - u$, show that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$. **2**

(ii) Hence evaluate $\int_0^2 x\sqrt{2-x}dx$. **2**

(b) Express $\dfrac{3\sqrt{3}+i}{\sqrt{3}-i}$ in the form $x+iy$, where x and y are real. **2**

(c) Find $\int e^x \cos x \, dx$. **2**

(d) Find the square roots of $1+\sqrt{3}i$. **2**

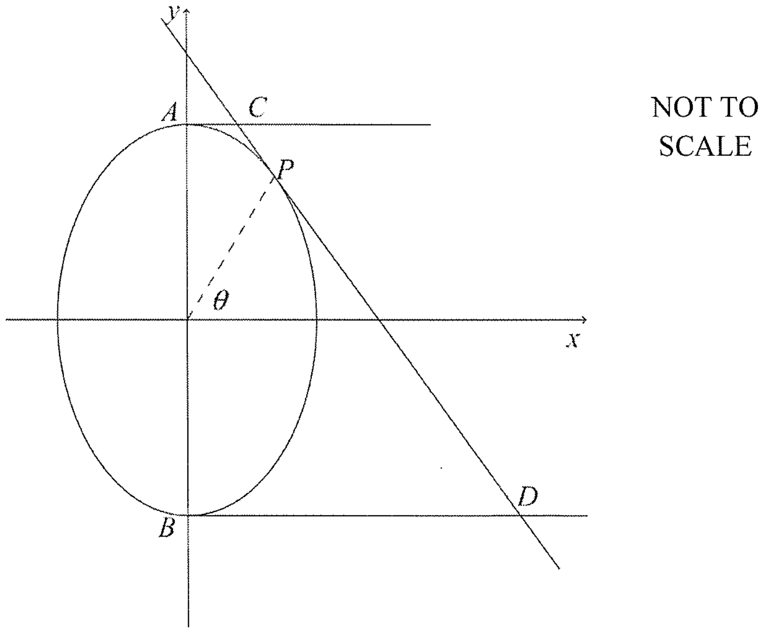
(e) Given that α, β and γ are the roots of $x^3+px^2+qx+r=0$, find the equation whose roots are α^2, β^2 and γ^2. **2**

(f) Sketch the region in the complex plane where both the inequalities $|z-2-2i|<2$ and $0<\arg(z-2-2i)<\dfrac{\pi}{4}$ hold true simultaneously. **3**

Question 12 (15 marks) Start a new writing booklet

(a) Find $\int \frac{1}{8+5\sin x}\,dx$. **2**

(b) The diagram below shows the ellipse which has equation $\frac{x^2}{4}+\frac{y^2}{9}=1$. The point $P(2\cos\theta, 3\sin\theta)$, where θ is the axillary angle, lies on the ellipse. The ellipse meets the y-axis at the points A and B. The tangents to the ellipse at A and B meet the tangent at P at the points C and D respectively.



(i) Find the eccentricity, coordinates of the foci and the equation of the directricies. **3**

(ii) Show that the equation of the tangent to the ellipse at P is $2y\sin\theta + 3x\cos\theta = 6$. **2**

(iii) Find the numerical value of $AC \times BD$. **3**

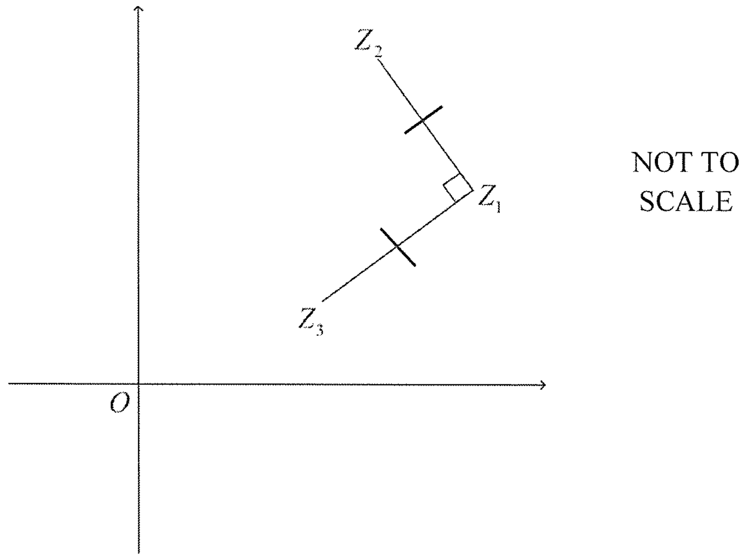
(c) For every integer $n \geq 0$, let $I_n = \int_0^{\frac{\pi}{6}} \sec^n x\,dx$. **3**

Show that for $n \geq 2$, $(n-1)I_n = \frac{2^{n-2}}{\left(\sqrt{3}\right)^{n-1}} + (n-2)I_{n-2}$.

Question 12 continues on page 9

Question 12 (continued)

(d)



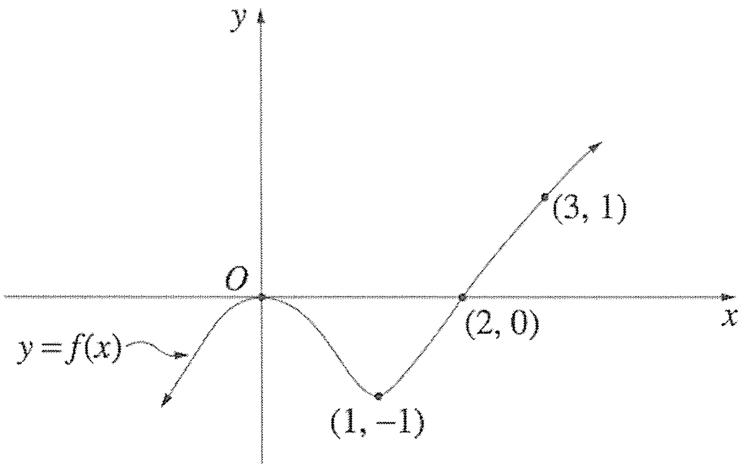
On the Argand diagram above, the point Z_1 represents the complex number z_1 and the point Z_2 represents the complex number z_2. The point Z_2 is rotated about Z_1 through a right angle in the positive direction to take up the position Z_3, representing the complex number z_3. **2**

Show that $z_3 = (1-i)z_1 + iz_2$.

End of Question 12

Question 13 (15 marks)   Start a new writing booklet

(a) The diagram below shows the graph of $y = f(x)$.



Draw separate one-third page sketches of the graphs of the following:

(i) $y = \dfrac{1}{f(x)}$. **2**

(ii) $y = |f(x)|$. **2**

(iii) $y = \ln\left(f(x)\right)$. **2**

(b) (i) By using De Moivre's Theorem, show that $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ and $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$. **2**

(ii) Hence show that $\tan 3\theta = \dfrac{3t - t^3}{1 - 3t^2}$, where $t = \tan\theta$. **1**

(iii) Hence find the general solutions of the equation $3\tan\theta - \tan^3\theta = 0$ **1**

(c) (i) Find the five roots of the equation $z^5 = 1$ **2**

(ii) Show that $z^5 - 1 = (z-1)\left(z^2 - 2z\cos\dfrac{2\pi}{5} + 1)\right)\left(z^2 - 2z\cos\dfrac{4\pi}{5} + 1\right)$. **2**

(iii) Hence show that $\cos\dfrac{2\pi}{5} + \cos\dfrac{4\pi}{5} = -\dfrac{1}{2}$ **1**

Question 14 (15 marks) Start a new writing booklet

(a) Find $\int \frac{x+3}{x^3+x^2+x+1}\,dx$. **2**

(b) The base of a solid is the circle $x^2+y^2=36$. Find the volume of the solid if every section perpendicular to the x-axis is a square where one side of the square is completely laid in the base of the solid. **3**

(c) A parachutist of mass m falls to the ground from a plane. Air resistance is proportional to mv^2, where v is his speed and g is acceleration due to gravity. Take downwards as being the positive direction, and the point where the parachutist jumps out the plane as the origin of displacement, x.

(i) Deduce that $\frac{d}{dx}\left(v^2\right)=2g-2kv^2$, where k is the constant of proportionality. **1**

(ii) Show that $v^2=\frac{g}{k}-\frac{g}{k}e^{-2kx}$, satisfies the differential equation in part (i). **2**

(iii) Find an expression for the terminal speed of the parachutist during his free-fall. **1**

(d) Let $f(x)=3x^5-10x^3+16x$.

(i) Show that $f'(x)\geq 1$ for all real x. **2**

(ii) For what values of x is $f''(x)>0$. **2**

(iii) Sketch the graph of $y=f(x)$, clearly indicating any turning points and points of inflexion. **2**

Question 15 (15 marks) Start a new writing booklet

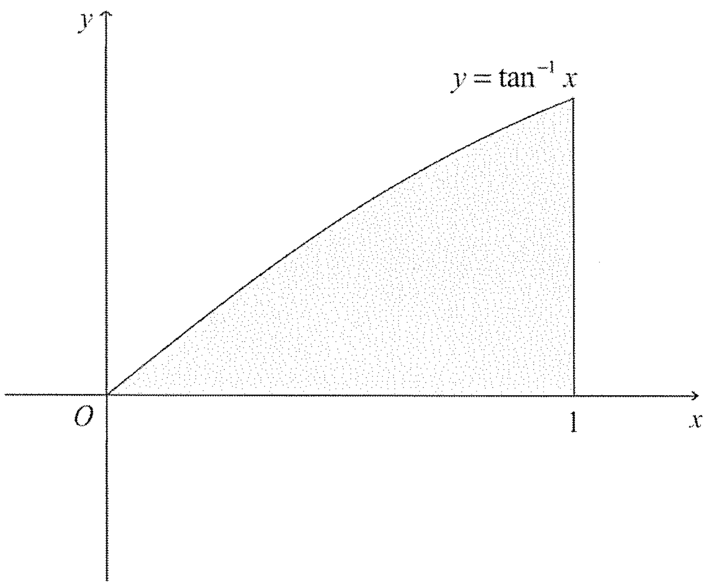
(a) Consider the function $f(u)=\sin^{-1}u-\sqrt{1-u^2}$, with restricted domain $0<u<1$.

(i) Show that $f'(u)=\sqrt{\dfrac{1+u}{1-u}}$. **1**

(ii) Hence, given that α is in the domain, show that **2**

$$\int_0^{\alpha}\left(\frac{1+u}{1-u}\right)^{\frac{1}{2}}du=\sin^{-1}\alpha-\sqrt{1-\alpha^2}+1$$

(b) The region bounded by the curve $y=\tan^{-1}x$ and the x axis between $x=0$ and $x=1$ is rotated through one complete revolution about $x=1$. A diagram of the region to be rotated is shown below.



NOT TO SCALE

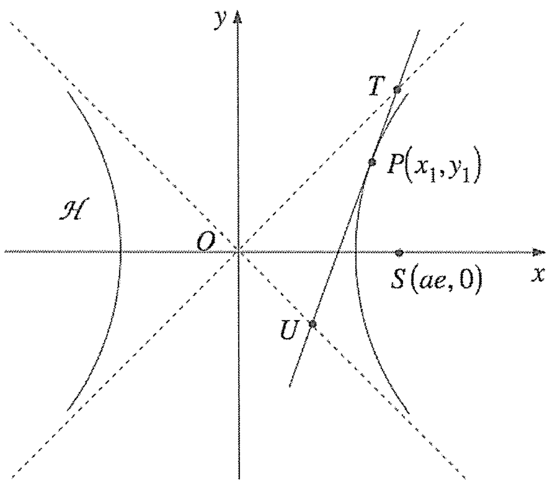
(i) Use the method of cylindrical shells to show that the volume V of the solid formed is given by $V=2\pi\int_0^1(1-x)\tan^{-1}x\,dx$. **1**

(ii) Hence find the volume V in simplest exact form. **4**

Question 15 continues on page 13

Question 15 (continued)

(c) The point $S(ae, 0)$ is a focus on the hyperbola $H: x^2 - y^2 = a^2$. The tangent to the hyperbola at a point $P(x_1, y_1)$ meets the asymptotes of H at T and U, as shown in the diagram below.



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(i) Show that the equation of the tangent TU is $x_1 x - y_1 y = a^2$. **2**

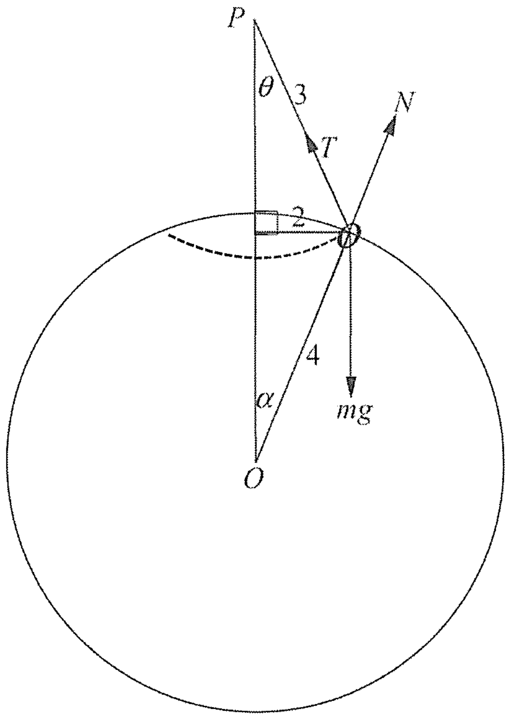
(ii) Show that the gradient of SU is $\dfrac{a}{e(x_1 + y_1) - a}$. **2**

(iii) Let $\angle UST = \theta$. Show that $\tan\theta = -1$. **3**

End of Question 15

Question 16 (15 marks) Start a new writing booklet

(a) A particle of mass 5 kg at the end of a string 3 metres long is suspended from a point P vertically above the highest point of a smooth sphere of radius 4 metres. It describes a horizontal circle of radius 2 metres on the surface of the sphere.



NOT TO SCALE

Three forces act on the particle: the tension force F of the string, the normal reaction force N to the surface of the sphere, and the gravitational force mg. Take g, the acceleration due to gravity, as $10\ \text{ms}^{-2}$. The angular velocity of the particle moving in uniform circular motion is 1 radian per second.

(i) By resolving the forces horizontally and vertically on a diagram, show that **2**

$$\frac{T\sqrt{5}}{3}+\frac{N\sqrt{3}}{2}=50 \quad \text{and} \quad \frac{2T}{3}-\frac{N}{2}=10\,.$$

(ii) Find, correct to one decimal place:

(α) the tension in the string. **1**

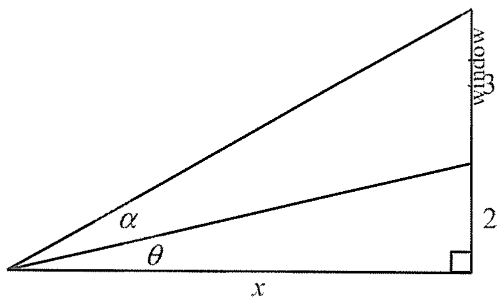
(β) The force exerted on the sphere. **1**

(iii) Find the angular velocity that will ensure there is no force exerted on the sphere. **1**

Question 16 continues on page 15

Question 16 (continued)

(b) The base of a stained glass window 3 metres high is 2 metres above the eye-level of an observer who is x metres from the base of the wall which is supporting the window. α is the viewing angle at eye level (i.e. the difference between the angles of elevation of the top and bottom of the window, as seen by the observer)



NOT TO SCALE

(i) Show that $\alpha = \tan^{-1}\left(\dfrac{3x}{x^2+10}\right)$. **3**

(ii) Hence find how far should the observer stand from the wall for the viewing angle to be greatest. **3**

(c) Given that $f(x) = x^6 + 4x^5 - 3x^4 - 8x^3 + 35x^2 - 60x - 225$ has zeroes at $x = \pm\sqrt{5}$ and a double zero, factorise $f(x)$ over the:

(i) real field. **3**

(ii) complex field. **1**

End of Paper

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