HORNSBY GIRLS HIGH SCHOOL

2011
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- **Reading Time – 5 minutes**
- **Working Time – 3 hours**
- **Write using a black or blue pen**
- **Approved calculators may be used**
- **A table of standard integrals is provided at the back of this paper**
- **All necessary working should be shown for every question**
- **Begin each question in a new booklet**

Total marks (120)

- **Attempt Questions 1 – 8**
- **All questions are of equal value**

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Total Marks
Attempt Questions 1–8
All Questions are of equal value

Begin each question in a new booklet, writing your student number and question number in the boxes indicated. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find $\int \frac{x^2}{\sqrt{8-x^3}}\,dx$. **2**

(b) By completing the square, find $\int \frac{dx}{x^2-8x+20}$. **2**

(c) Evaluate $\int_0^{\pi} x\cos x\,dx$. **3**

(d) (i) Show that $\frac{2}{x^3+x^2+x+1} = \frac{1}{x+1} - \frac{x}{x^2+1} + \frac{1}{x^2+1}$. **2**

(ii) Hence, or otherwise, show $\int_{\frac{1}{2}}^{2} \frac{2}{x^3+x^2+x+1} = \tan^{-1}2 - \tan^{-1}\left(\frac{1}{2}\right)$. **3**

(e) Using the substitution $x = \tan\theta$, or otherwise, show **3**

$$\int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{1+x^2}}\,dx = \sqrt{2} - \frac{2}{\sqrt{3}}.$$

Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Write i^7 in the form $x+iy$ where x and y are real. **1**

(b) Let $z=2+2i$ and $w=2-i$. Find in the form $x+iy$, where x and y are real,

(i) $z\overline{w}$ **1**

(ii) $\frac{8}{z}$ **1**

(c) It is given that $1+i$ is a root of $P(z)=2z^3-3z^2+rz+s$, where r and s are real.

(i) Explain why $1-i$ is also a root of the equation. **1**

(ii) Factorise $P(z)$ over the real field. **2**

(d) Find all the solutions of $z^4=16$. Express your solutions in the modulus-argument form. **2**

(e) Sketch the region in the complex plane where the inequalities $|z-\overline{z}|\leq 2$ and $|z-i|\leq 4$ hold. **3**

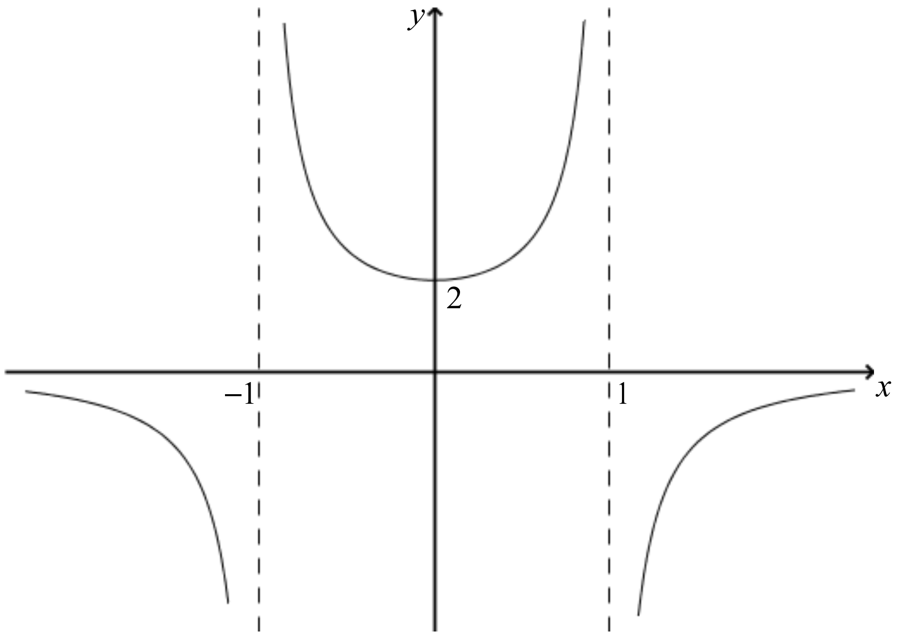
(f) (i) Prove, by Mathematical Induction, that for all integers n, **3**

$$(\cos\theta+i\sin\theta)^n=\cos(n\theta)+i\sin(n\theta)$$

(ii) Hence, find an expression for $\cos 3\theta$. **1**

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The diagram shows the graph of $y = f(x)$. The graph has a horizontal asymptote at $y = 0$ and vertical asymptotes at $x = \pm 1$.



NOT TO SCALE

Draw neat separate one-third page sketches of the graphs of the following:

(i) $y = \dfrac{1}{f(x)}$ **2**

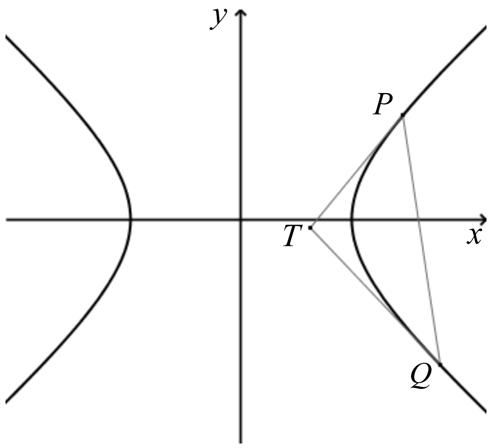
(ii) $y = f(x) + |f(x)|$ **2**

(iii) $y = e^{f(x)}$ **2**

Question 3 continues on page 6

Question 3 (continued)

(b)



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The points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the right branch of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
The tangents at P and Q meet at $T(x_0, y_0)$.

(i) Show the equation of the tangent at P is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ **2**

(ii) Hence show the equation of the chord of contact is $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$. **2**

(iii) The chord PQ passes through the focus $S(ae, 0)$ where e is the eccentricity of the hyperbola. Prove T lies on the directrix of the parabola. **1**

(c) Let α, β, γ be the zeros of the polynomial $P(x) = 3x^3 + 7x^2 + 11x + 51$.

(i) Find $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$. **1**

(ii) Find $\alpha^2 + \beta^2 + \gamma^2$. **2**

(iii) Using part (ii), or otherwise, determine how many zeros of $P(x)$ are real. Justify your answer. **1**

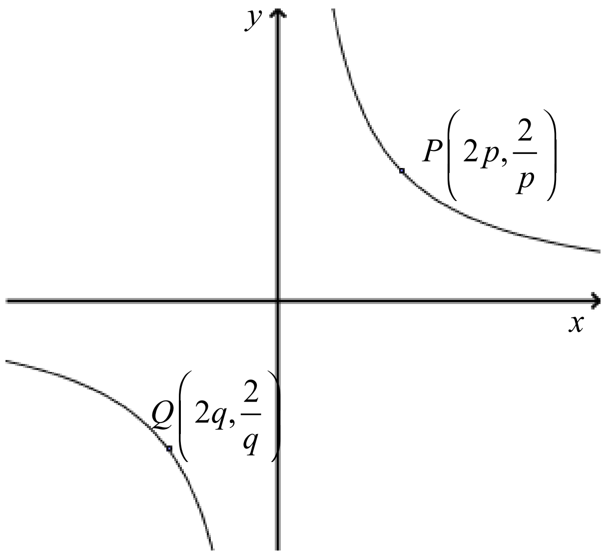
Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) A solid of height 2 metres rests on a horizontal surface. **3**

Every horizontal cross-section of the solid, x metres above the surface, is a square of side $\sqrt{3x+1}$ metres.

Find the volume of the solid.

(b) Consider the rectangular hyperbola $xy = 4$, with points P and Q on different branches of the hyperbola.



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(i) Prove that the equation of the normal to $xy = 4$ at the point $P\left(2p, \frac{2}{p}\right)$ is $py - p^3x = 2(1-p^4)$. **3**

(ii) If this normal meets the hyperbola again at $Q\left(2q, \frac{2}{q}\right)$, prove that $q = \frac{-1}{p^3}$. **2**

(iii) Hence, show that there exists only one chord of the hyperbola which is normal to the hyperbola at P and Q, and find its equation. **3**

(c) The equation $x^3 + 3x + 2 = 0$ has roots α, β and γ.

(i) Find the polynomial whose roots are α^2, β^2 and γ^2. **2**

(ii) Hence, or otherwise, find the value of $\alpha^3 + \beta^3 + \gamma^3$. **2**

Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let w be a complex root of unity (w is solution of $z^3 - 1 = 0$).

(i) Show that $(z-1)(z^2+z+1) = z^3 - 1$. **1**

(ii) Explain why $w^2 + w + 1 = 0$. **1**

(iii) Hence, other otherwise, show that $(1-w)(1-w^2)(1-w^4)(1-w^8) = 9$ **3**

(b) Consider $I = \int_1^{\infty} \frac{1}{x\sqrt{1+x^2}}\, dx$.

(i) By using a suitable substitution, show that $I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin\theta} d\theta$. **2**

(ii) Hence, or otherwise, evaluate I. **3**

(c) (i) Find real numbers, a and b, such that **2**

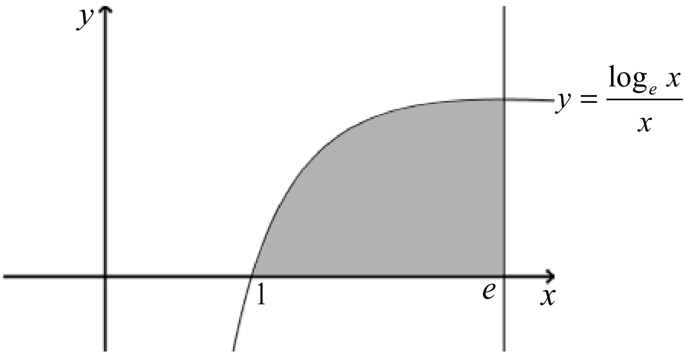
$x^4 + x^3 + x^2 + x + 1 = (x^2 + ax + 1)(x^2 + bx + 1)$.

(ii) Given that $x = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$ is a solution of $x^4 + x^3 + x^2 + x + 1 = 0$,

find the exact value of $\cos\frac{2\pi}{5}$. **3**

Question 6 (15 marks) Use a SEPERATE writing booklet. **Marks**

(a) Use the method of cylindrical shells to find the volume of the solid formed when the shaded region bounded by $y=0$, $y=\dfrac{\log_e x}{x}$ and $x=e$ is rotated about the y-axis. **4**



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(b) (i) Show that $\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta$. **1**

(ii) Hence, or otherwise, solve the equation **3**

$\sin\theta+\sin 2\theta+\sin 3\theta+\sin 4\theta=0$ for $0\le\theta\le 2\pi$.

(c) A stone is projected vertically upwards in the air from a point h metres above the ground at a speed u and experiences a resistance equal to mkv^2, where m is the mass of the stone, v is the speed after time t and k is a constant. **3**

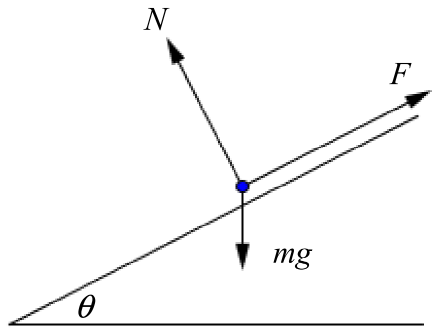
By considering the forces acting on the stone, show that the maximum height, H, the stone reaches above the ground is given by $H=h+\dfrac{1}{2k}\ln\left(1+\dfrac{ku^2}{g}\right)$, where g is acceleration due to gravity

(d) A group of n people are to be seated around a circular table. Find the number of possible arrangements if 3 particular people are to sit together. **2**

(e) Show that ${}^{n+2}C_r={}^{n}C_r+2\,{}^{n}C_{r-1}+{}^{n}C_{r-2}$ **2**

Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a)



A particle of mass m is lying on an inclined plane and does not move. **2**
The plane is at an angle θ to the horizontal. The particle is subject to a gravitational force mg, a normal reaction force N, and a frictional force F parallel to the plane, as shown in the diagram above.

By resolving the forces acting on the particle parallel and perpendicular to the plane, find an expression for $\frac{F}{N}$ in terms of θ.

(b) The polynomial $P(x) = x^4 - 4x^3 + 3x^2 - 14x + 10$ has roots $a + ib$, $a - 2ib$, where a and b are real.

(i) Show that $a = 1$, and hence find the value(s) of b. **2**

(ii) Hence, factorise $P(x)$ over the rational field. **2**

(c) (i) If $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$, then show that $I_n = \frac{n-1}{n} I_{n-2}$. **3**

(ii) Hence, evaluate $\int_0^{\frac{\pi}{2}} \cos^5 x \, dx$. **2**

(d) Use Mathematical Induction to prove that for integer values of $n \geq 1$ **4**

$1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + n \times n! = (n+1)! - 1$

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The function $y = f(x)$ is defined by $f(x) = \sqrt{3 - \sqrt{x}}$

(i) State the domain of the function $f(x)$. **1**

(ii) Show that $y = f(x)$ is a decreasing function and determine the range of $y = f(x)$. **2**

(iii) Sketch the graph of $y = f(x)$ for the domain and range determined above. **1**

(iv) Prove that $\int_0^9 \sqrt{3 - \sqrt{x}}\, dx = \frac{24\sqrt{3}}{5}$ **2**

(b) Show that $\frac{d}{du}\left(\sec u + \tan u\right) = \sec u(\sec u + \tan u)$ **2**

(c) Consider $f(x) = \cos\frac{x}{2}$.

(i) On the same set of axes, sketch the graph of $y = f(x)$, and hence the graph of $y = \frac{1}{f(x)}$ for the domain $0 \le x \le 2\pi$. **2**

(ii) By considering part (b), find the area bounded by the curve $y = \frac{1}{f(x)}$, the $x-$axis and the ordinates $x = \frac{\pi}{3}$ and $x = \frac{\pi}{2}$, leaving your answer exact. **3**

(iii) The solid bounded by the curve $y = \frac{1}{f(x)}$, the $x-$axis and the ordinates $x = \frac{\pi}{3}$ and $x = \frac{\pi}{2}$ is rotated about the $y-$axis. **2**

By using the method of annular discs, find the volume as a **definite integral**.

DO NOT EVALUATE THIS INTEGRAL.

End of paper