HORNSBY GIRLS' HIGH SCHOOL

2007
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using a black or blue pen
- Approved calculators may be used
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown for every question.
- Begin each question on a fresh sheet of paper.

Total marks (120)

- Attempt Questions 1-8
- All questions are of equal value

Total Marks – 120
Attempt Questions 1-8
All Questions are of equal value
Begin each question on a NEW SHEET of paper, writing your student number and question number at the top of the page. Extra paper is available.

Question 1 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) Use the technique of integration by parts to find:

(i) $\int \ln x \, dx$ **2**

(ii) $\int e^x \cos x \, dx$ **3**

(b) Use partial fractions to find $\int \frac{4dx}{4x^2-1}$ **2**

(c) Find $\int \frac{dx}{x^2+2x+4}$ **2**

(d) Find $\int \sqrt{\frac{x-1}{x+1}} \, dx$ **2**

(e) By using the substitution $t = \tan\left(\frac{x}{2}\right)$ and partial fractions evaluate $\int_0^{\pi/2} \frac{dx}{4\sin x + 3\cos x}$ **4**

Question 2 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) Given that P and Q represent the complex numbers $5+2\sqrt{6}i$ and $1-\sqrt{3}i$ respectively, find:

(i) $\frac{P}{Q}$ in the form $x+iy$ **2**

(ii) $\overline{P} \times \overline{Q}$ **2**

(iii) $\sqrt{P}$ in the form $x+iy$ **2**

(iv) The modulus and argument of Q **2**

(v) The complex number R in the form $x+iy$, given that $\arg R = 2\arg Q$ and $|R| = 2|Q|$ **2**

(b) On an Argand diagram sketch the region defined by $-2 \le \text{Re}(Z) < 1$ **1**

(c) Draw a sketch in the complex plane of the locus of Z given by the equations

(i) $\arg(Z-3+2i) = \frac{\pi}{4}$ **2**

(ii) $\arg(Z-1) - \arg(Z+1) = \frac{\pi}{2}$ **2**

Question 3 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) Given $f(x) = e^x - 2$ draw large (half page), separate, neat and accurate sketches of each of the following, showing clearly all the intercepts and asymptotes:

(i) $y = f(x)$ **2**

(ii) $y = |f(x)|$ **2**

(iii) $y = \frac{1}{f(x)}$ **2**

(iv) $y^2 = f(x)$ **2**

(b) The region bounded by the curve $y = x^2 - 4x + 4$ and the x and y axes is rotated about the line $y = -1$. Find the volume of the solid of revolution. **4**

(c) An ellipse has equation $\frac{x^2}{4} + \frac{y^2}{9} = 1$. Find the eccentricity, co-ordinates of the foci S and S' and the equations of the directrices. **3**

Question 4 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) Find:

(i) $\int \sin^3 x . \cos^5 x \, dx$ **3**

(ii) $\int \frac{dx}{x^3\sqrt{x^2 - 4}}$ **3**

(iii) $\int \tan^4 x \, dx$ **3**

(b) (i) Show that a reduction formula for, $I_n = \int x^n \cos x \, dx,$ is

$$I_n = x^n \sin x + nx^{n-1} \cos x - n(n-1)I_{n-2.}$$ **3**

(ii) Hence, or otherwise, evaluate $\int_0^{\pi/2} x^4 \cos x \, dx$ **3**

Question 5 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) A mass of 3 *kg*, on the end of a string 0.8 metres long, is rotating as a conical pendulum with angular velocity 3π radians per second. Use $g = 10m/s^2$ and let θ be the angle that the string makes with the vertical.

(i) Draw a diagram showing all the forces acting on the mass **1**

(ii) By resolving forces, find the tension in the string **2**

(iii) Find θ correct to the nearest degree **1**

(b) A particle is dropped from rest at a height h metres above the ground. At time t seconds its height above the ground is given by

$$x = h + \frac{gt}{k} + \frac{ge^{-kt}}{k^2} - \frac{g}{k^2}$$

(i) Show that $\ddot{x} = g - kv$ where the velocity of the particle is v *m/s* **2**

(ii) What forces are acting on this particle ? Explain carefully. **1**

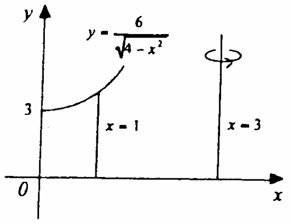
(iii) If it takes T seconds for the particle to reach half its terminal velocity, find the value of e^{kT}. **2**

(c) Find the magnitude of the braking force required to stop a truck of mass 4800 *kg* in 55 metres when it is traveling at 40 *km/h* down an incline of angle 5^0 to the horizontal. (assume no wind resistance and use $g = 10$ *m/s²*) **3**

(d) Prove the identity $\frac{\cos y - \cos(y + 2x)}{2\sin x} = \sin(y + x)$ **3**

Question 6 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a)



A mould for a section of concrete piping is made by rotating the region bounded by the curve $y = \dfrac{6}{\sqrt{4-x^2}}$ and the x-axis between the lines $x = 0$ and $x = 1$ through one complete revolution about the line $x = 3$. All measurements are in metres.

(i) By considering strips of width δx parallel to the axis of rotation, show that the volume $V\,m^3$ of the concrete used in the piping is given by $V = 12\pi \displaystyle\int_0^1 \frac{3-x}{\sqrt{4-x^2}}\,dx$ **3**

(ii) Hence, or otherwise, find the volume of the concrete used in the piping, giving your answer correct to the nearest cubic metre. **3**

(b) (i) Sketch the graph of the curve $y = x + e^{-x}$ showing clearly the coordinates of any turning points and the equations of any asymptotes. **2**

(ii) The region in the first quadrant between the curve $y = x + e^{-x}$ and the line $y = x$ and bounded by the lines $x = 0$ and $x = 1$ is rotated through one complete revolution about the y-axis. Use the method of cylindrical shells to find the volume of the solid. **5**

(c) The expression $\sqrt{12 + \sqrt{12 + \sqrt{12 + \sqrt{12 + \sqrt{\ldots\ldots\ldots}}}}}$ has a limit L.
Find the exact value of L. **2**

Question 7 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) The roots of $px^3 + qx^2 + rx + s = 0$ form a geometric series. Prove that $pr^3 = q^3s$ **3**

(b) If i is a root of $z^4 + 2z^3 - 2z^2 + 2z - 3 = 0$, find the other three roots. **3**

(c) Given that $Q(x) = x^4 - 5x^3 + 4x^2 + 3x + 9$ has a zero of multiplicity 2, solve the equation $Q(x) = 0$ over the complex field. **3**

(d) Given the function $f(x) = \sqrt{2 - \sqrt{x}}$

(i) What is the domain of $f(x)$? **1**

(ii) Show that $f(x)$ is a decreasing function and deduce the range of $f(x)$. **2**

(iii) By considering the graph of $y = f(x)$, or otherwise, evaluate $\int_0^4 \sqrt{2 - \sqrt{x}}\,dx$ **3**

Question 8 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) Consider the rectangular hyperbola $xy = 4$

(i) Show that the gradient of the tangent at the point $P\left(2p, \frac{2}{p}\right)$ is $\frac{-1}{p^2}$ **1**

(ii) Show that the equation of the normal at P is given by $p^3x - py = 2(p^4 - 1)$ **1**

(iii) This normal meets the hyperbola again at $Q\left(2q, \frac{2}{q}\right)$. Prove that $p^3q = -1$. **3**

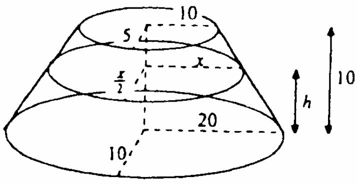
(iv) Hence, or otherwise, find the equation of the chord that is a normal at both ends of the chord. **2**

(b) The line $3y = 5x + 1$ is the equation of the diagonal of a square. One of the square's vertices is (3,11). Find the coordinates of the other vertices. **3**

(c) A solid of height 10 metres stands on horizontal ground. The base of the solid is an ellipse with semi-axes 20 metres and 10 metres.
Horizontal cross-sections taken parallel to the base and at height h metres above the base are ellipses with semi-axes x metres and $\frac{x}{2}$ metres.
The centres of these elliptical cross-sections and the base lie on a vertical straight line, and the extremities of their semi-axes lie on sloping straight lines as shown in the diagram. The top of the solid is an ellipse with semi-axes 10 metres and 5 metres.

Find the volume of the solid correct to the nearest cubic metre.

(you may assume that the area contained by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab). **5**



END OF PAPER

Ext 2 Trial 2007 Solutions

a) i) $\int \ln x \, dx$ let $u = \ln x$ $v = x$

$\frac{du}{dx} = \frac{1}{x}$ $\frac{dv}{dx} = 1$

$= x\ln x - \int x . \frac{1}{x} dx$

$= x \ln x - \int 1 \, dx$

$= x \ln x - x \; (+C)$

) $I = \int e^x \cos x \; dx$ let $u = \cos x$ $v = e^x$

$\frac{du}{dx} = -\sin x$ $\frac{dv}{dx} = e^x$

$I = e^x \cos x - -\int e^x \sin x \, dx$

$= e^x \cos x + \int e^x \sin x \, dx$ $u = \sin x$ $v = e^x$

$\frac{du}{dx} = \cos x$ $\frac{dv}{dx} = e^x$

$= e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$

$= e^x(\cos x + \sin x) - I$

$\therefore 2I = e^x(\cos x + \sin x)$

$I = \frac{e^x}{2}(\cos x + \sin x)$

) let $\frac{4}{4x^2-1} = \frac{4}{(2x-1)(2x+1)} = \frac{A}{2x-1} + \frac{B}{2x+1}$

$\therefore 4 = A(2x+1) + B(2x-1)$

let $x = -1/2$, $4 = -2B$

$B = -2$

let $x = 1/2$ $4 = 2A$

$A = 2$

$\therefore \int \frac{4}{4x^2-1} dx = \int \left(\frac{2}{2x-1} - \frac{2}{2x+1}\right) dx$

$= \ln(2x-1) - \ln(2x+1) \; (+C)$

$= \ln\left(\frac{2x-1}{2x+1}\right) \; (+C)$

c) $\int \frac{dx}{x^2+2x+4} = \int \frac{dx}{(x+1)^2+3}$

$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)$

) $\int \sqrt{\frac{x-1}{x+1}} \, dx = \int \frac{x-1}{\sqrt{x^2-1}} dx$

$= \frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}} dx - \int \frac{dx}{\sqrt{x^2-1}}$

$= \sqrt{x^2-1} - \ln\left(x + \sqrt{x^2-1}\right)$

Q1e) $\int_0^{\frac{\pi}{2}} \frac{dx}{4\sin x + 3\cos x}$

$t = \tan\left(\frac{x}{2}\right)$

$\sin x = \frac{2t}{1+t^2}$ $\cos x = \frac{1-t^2}{1+t^2}$

$\frac{dt}{dx} = 1/2 \sec^2\left(\frac{x}{2}\right) = \frac{1}{2}\left(1+\tan^2\left(\frac{x}{2}\right)\right)$

$\therefore \frac{dx}{dt} = \frac{2}{1+t^2}$

$x = 0, t = 0$ $x = \frac{\pi}{2}, t = 1$

$= \int_0^1 \frac{1}{\frac{8t}{1+t^2} + \frac{3(1-t^2)}{1+t^2}} \cdot \frac{2dt}{1+t^2}$

$= \int_0^1 \frac{2dt}{3+8t-3t^2}$

$= \int_0^1 \frac{2 \, dt}{(3-t)(1+3t)}$

let $\frac{2}{(3-t)(1+3t)} = \frac{A}{3-t} + \frac{B}{1+3t}$

$\therefore 2 = A(1+3t) + B(3-t)$

let $t = 3$, $2 = 10A$

$A = \frac{1}{5}$

$t = -1/3$ $2 = \frac{10B}{3}$

$B = \frac{3}{5}$

$\therefore \int_0^1 \frac{2dt}{(3-t)(1+3t)} = \int_0^1 \left(\frac{\frac{1}{5}}{(3-t)} + \frac{\frac{3}{5}}{(1+3t)}\right) dt$

$= \left[-\frac{1}{5}\ln(3-t) + \frac{1}{5}\ln(1+3t)\right]_0^1$

$= \frac{1}{5}\left[\ln\left(\frac{1+3t}{3-t}\right)\right]_0^1$

$= \frac{1}{5}\left[\ln\left(\frac{4}{2}\right) - \ln\left(\frac{1}{3}\right)\right]$

$= \frac{1}{5} \ln 6$

Q2a) i) $\frac{P}{Q} = \frac{5+2\sqrt{6}i}{1-\sqrt{3}i} \times \frac{1+\sqrt{3}i}{1+\sqrt{3}i}$

$= \frac{5+5\sqrt{3}i+2\sqrt{6}i-2\sqrt{18}}{4}$

$= \frac{5-6\sqrt{2}}{4} + i\left(\frac{5\sqrt{3}+2\sqrt{6}}{4}\right)$

ii) $\bar{P} \times \bar{Q} = (5-2\sqrt{6}i)(1+\sqrt{3}i)$

$= 5+5\sqrt{3}i-2\sqrt{6}i+2\sqrt{18}$

$= 5+6\sqrt{2}+i(5\sqrt{3}-2\sqrt{6})$

ii) let $\sqrt{P} = x+iy$

$P = x^2 - y^2 + 2ixy$

$\therefore x^2 - y^2 = 5$ — ①

$2xy = 2\sqrt{6}$ — ②

from ② $y = \frac{\sqrt{6}}{x}$ — ③

Subst ③ into ① ⇒ $x^2 - \frac{6}{x^2} = 5$

$x^4 - 6 = 5x^2$

$x^4 - 5x^2 - 6 = 0$

$(x^2-6)(x^2+1) = 0$

$x = \pm\sqrt{6}$

$y = \pm 1$

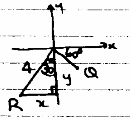
$\therefore \sqrt{P} = \sqrt{6}+i \ , -\sqrt{6}-i$

) Modulus : $|Q| = \sqrt{1+(\sqrt{3})^2} = 2$, $\arg Q = -60°$

) $\arg R = -120°$

$|R| = 4$

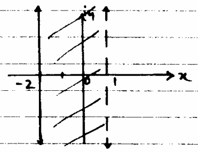
$\therefore R = -2-2\sqrt{3}i$



$\sin 30° = \frac{x}{4}$ $\cos 30° = \frac{y}{4}$

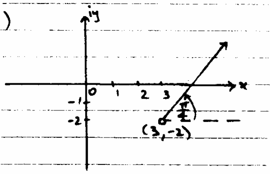
$x = 2$ $y = 2\sqrt{3}$

)

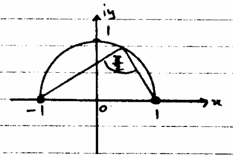


$-2 \leq Re(z) < 1$

) i)



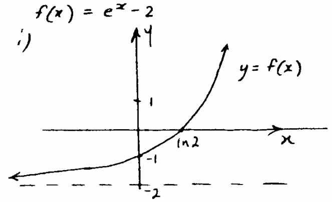
ii)



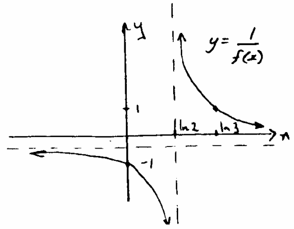
QUESTION 3

$f(x) = e^x - 2$

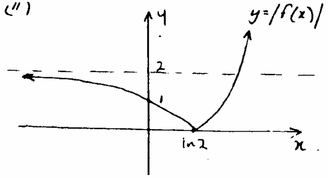
a) i)



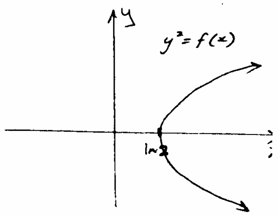
(iii)



(ii)



(iv)



b) $V = \lim_{\Delta x \to 0} \sum_0^2 (\pi R^2 - \pi r^2)\Delta x$

$= \pi \int_0^2 \left(1+(x-2)^2-1\right)\left(1+(x-2)^2+1\right) dx$.

$= \pi \int_0^2 (x-2)^2\left(2+(x-2)^2\right) dx$.

$= \pi \int_0^2 2(x-2)^2 + (x-2)^4 \ dx$

$= \pi \left[\frac{2(x-2)^3}{3} + \frac{(x-2)^5}{5} \right]_0^2$

$= \frac{176\pi}{15}$

c) $a^2 = 4$ $b^2 = 9$

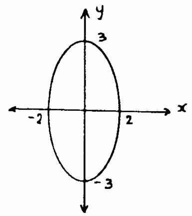
$a = 2$ $b = 3$

$a^2 = b^2(1-e^2)$

$4 = 9(1-e^2)$

$e^2 = 5/9$

$e = \frac{\sqrt{5}}{3}$



foci $S(0,\sqrt{5})$

$S'(0,-\sqrt{5})$

directrices $y = \pm 9$

QUESTION 4

a) i) $I = \int \sin^3 x . \cos^5 x \, dx.$

$= \int \sin^3 x \left(1 - \sin^2 x\right)^2 . \cos x \, dx$

$= \int \sin^3 x \left(1 - 2\sin^2 x + \sin^4 x\right) \cos x \, dx.$

$= \int \left(\sin^3 x - 2\sin^5 x + \sin^7 x\right) \cos x \, dx.$

let $u = \sin x$

$du = \cos x \, dx$

$\therefore I = \int \left(u^3 - 2u^5 + u^7\right) du$

$= \frac{u^4}{4} - \frac{2u^6}{6} + \frac{u^8}{8}$

$\therefore I = \frac{1}{4}\sin^4 x - \frac{1}{3}\sin^6 x + \frac{1}{8}\sin^8 x$

OR $I = \frac{1}{8}\cos^8 x - \frac{1}{6}\cos^6 x$

Method 2

$I = \int \left(1 - \cos^2 x\right) \cos^5 x . \sin x \, dx$

$= \int \left(\cos^5 x - \cos^7 x\right). \sin x \, dx$

$= \int \left(\cos^7 x - \cos^5 x\right). -\sin x \, dx$

let $u = \cos x$

$\frac{du}{dx} = -\sin x$

$= \int u^7 - u^5 \, du$

$= \frac{u^8}{8} - \frac{u^6}{6} + c$

$= \frac{\cos^8 x}{8} - \frac{\cos^6 x}{6} + c$

ii) $I = \int \frac{dx}{x^3\sqrt{x^2-4}}$

let $x = 2\sec\theta$

$dx = 2\sec\theta\tan\theta \, d\theta$

$\therefore I = \int \frac{2\sec\theta\tan\theta \, d\theta}{8\sec^3\theta\sqrt{4\sec^2\theta - 4}}$

$= \int \frac{\tan\theta \, d\theta}{4\sec^2\theta . 2\tan\theta}$

$= \frac{1}{8}\int \frac{d\theta}{\sec^2\theta}$

$= \frac{1}{8}\int \cos^2\theta \, d\theta$

$= \frac{1}{16}\int 1 + \cos 2\theta \, d\theta$

$= \frac{1}{16}\left(\theta + \frac{1}{2}\sin 2\theta\right)$

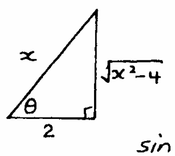
$= \frac{1}{32}\left(2\theta + \sin 2\theta\right) = \frac{1}{32}\left[2\cos^{-1}\frac{2}{x} + 4\frac{\sqrt{x^2-4}}{x^2}\right]$ *

since $x = 2\sec\theta$

$\frac{x}{2} = \frac{1}{\cos\theta}$

$\therefore \cos\theta = \frac{2}{x}$

$\theta = \cos^{-1}\left(\frac{2}{x}\right)$



$\sin 2\theta = 2\sin\theta\cos\theta$

$= 2 . \frac{\sqrt{x^2-4}}{x} . \frac{2}{x}$

$= \frac{4\sqrt{x^2-4}}{x^2}$

iii) $I = \int \tan^4 x \, dx$

$= \int \tan^2 x \tan^2 x \, dx$

$= \int \tan^2 x \left(\sec^2 x - 1\right) dx$

$= \int \sec^2 x \tan^2 x - \tan^2 x \, dx$

$= \int \sec^2 x \tan^2 x - \sec^2 x + 1$

$= \frac{1}{3}\tan^3 x - \tan x + x$

QUESTION 4

) i) $I_n = \int x^n \cos x \; dx$.

let $u = x^n \qquad dv = \cos x \; dx$
$du = nx^{n-1} dx \qquad V = \sin x$

$I_n = x^n \sin x - n \int x^{n-1} \sin x \; dx$.

let $u = x^{n-1} \qquad dv = \sin x \; dx$
$du = (n-1) x^{n-2} dx \qquad V = -\cos x$.

$I_n = x^n \sin x - n\left(-x^{n-1} \cos x + \int (n-1) x^{n-2} \cos x \; dx\right)$

$= x^n \sin x + nx^{n-1} \cos x - n(n-1) \int x^{n-2} \cos x \; dx$.

$= x^n \sin x + x^{n-1} \cos x - n(n-1) I_{n-2}$

ii) let $I_4 = \int x^4 \cos x \; dx$.

$I_0 = \int \cos x \; dx$
$= \sin x$
$I_2 = x^2 \sin x + 2x \cos x - 2 \sin x$.
$I_4 = x^4 \sin x + 4x^3 \cos x - 12\left(x^2 \sin x + 2x \cos x - 2 \sin x\right)$
$= x^4 \sin x + 4x^3 \cos x - 12x^2 \sin x + 24x \cos x + 24 \sin x$.

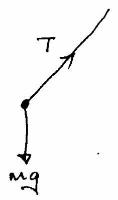
$\int_0^{\pi/2} x^4 \cos x \; dx = \left[x^4 \sin x + 4x^3 \cos x - 12x^2 \sin x + 24x \cos x + 24 \sin x\right]_0^{\pi/2}$

$= \left(\frac{\pi}{2}\right)^4 - 12\left(\frac{\pi}{2}\right)^2 + 24$

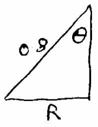
$= \frac{\pi^4}{16} - 3\pi^2 + 24$ #

QUESTION 5.

a) i)



ii)



$\sin\theta = \frac{R}{0.8}$

$R = 0.8 \sin\theta$

$T \sin\theta = m\omega^2 R$
$T \sin\theta = 3 \times 9\pi^2 \times 0.8 \sin\theta$
$T = 21.6\pi^2$
$= 213 \; N$

iii) $T \cos\theta = 30$
$\cos\theta = \frac{30}{T}$

$= 0.14$
$\theta = 82°$

b) $x = h + \frac{gt}{k} + \frac{ge^{-kt}}{k^2} - \frac{g}{k^2}$

$V = \dot{x} = \frac{g}{k} - \frac{ge^{-kt}}{k}$

$\ddot{x} = ge^{-kt}$

now $V = \frac{g}{k} - \frac{ge^{-kt}}{k}$.

⇗

$kv = g - ge^{-kt}$

$ge^{-kt} = g - kv$

$\therefore \ddot{x} = g - kv$

ii) $F = ma$
$= mg - mkv$
gravity and
resistance ∝ to velocity

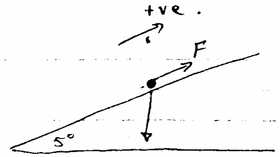
iii) $g = kV_T$
$V_T = \frac{g}{k}$

$\frac{V_T}{2} = \frac{g}{2k}$

$\therefore \frac{g}{2k} = \frac{g}{k} - \frac{ge^{-kT}}{k}$.

$\frac{1}{2} = 1 - e^{-kT}$
$e^{-kT} = \frac{1}{2}$
$e^{kT} = 2$.

)



$40 km/h = \frac{40000}{3600}$

$= \frac{100}{9}$ m/s .

$\frac{d(\frac{1}{2}v^2)}{dn} = a$

$\frac{1}{2}v^2 = ax + C$

$v^2 = 2an + 2C$

when $x = 0$, $V = -\frac{100}{9}$

$2C = \frac{10000}{81}$

$\therefore v^2 = 2ax + \frac{10000}{81}$

when $x = -55$, $V = 0$

$110a = \frac{10000}{81}$

$a = 1.12233$ m/s²

$\therefore$ $F - mg\sin 5° = 4800 \times 1.12233$

$F = 5387.205 + 48000\sin 5°$

$= 9570.68$ N

$= 9571$ N

d) $L.H.S. = \frac{\cos y - \cos(y+2x)}{2\sin x}$

$= \frac{\cos y - \cos y\cos 2x + \sin y\sin 2x}{2\sin x}$

$= \frac{\cos y - \cos y(\cos^2 x - \sin^2 x) + 2\sin y \overset{\sin x}{\wedge}\cos x}{2\sin x}$

$= \frac{\cos y(1 - \cos^2 x + \sin^2 x) + 2\sin y\sin x\cos x}{2\sin x}.$

$= \frac{\cancel{2}\sin^{\cancel{2}} x\cos y + 2\sin y\cancel{\sin x}\cos x}{\cancel{2}\cancel{\sin x}}$

$= \sin x\cos y + \sin y\cos x$

$= \sin(y+x)$

$= R.H.S.$

QUESTION 6

a) i) $\Delta V = \pi(R^2 - r^2)h$

$= \pi(R-r)(R+r)h$

$= \pi(3-x-3+x+\Delta x)(3-x+3-x-\Delta x)h$

$= \frac{6\pi}{\sqrt{4-x^2}}(6-2x-\Delta x)\Delta x$

$= \frac{6\pi}{\sqrt{4-x^2}}(6-2x)\Delta x$

$\therefore V = \lim_{\Delta x \to 0} \sum_{0}^{1} \Delta V$

$= 6\pi\int_0^1 \frac{6-2x}{\sqrt{4-x^2}}\,dx$

$= 12\pi\int_0^1 \frac{3-x}{\sqrt{4-x^2}}\,dx$

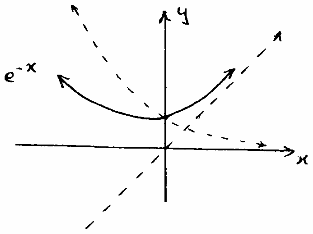
ii) $V = 12\pi\left(\int_0^1 \frac{3\,dx}{\sqrt{4-x^2}} + \frac{1}{2}\int_0^1 \frac{-2x\,dx}{\sqrt{4-x^2}}\right)$

$= 12\pi\left[3\sin^{-1}\frac{x}{2} + \sqrt{4-x^2}\right]_0^1$

$= 49$

b)

$y = x + e^{-x}$



turning point $(0,1)$
asymptote $y = x$

$\frac{dy}{dx} = 1 - e^{-x} = 0$

$e^{-x} = 1$

$x = 0$

$\therefore$ t.p. when $x = 0$, $y = 1$.

ii) $\Delta V = \pi(R^2 - r^2)h$

$= \pi((x+\Delta x)^2 - x^2)(x + e^{-x} - x))$

$= \pi(x^2 + 2x\Delta x + \Delta x^2 - x^2)e^{-x}$

$= \pi(2x\Delta x)e^{-x}$

$\therefore V = 2\pi\int_0^1 xe^{-x}\,dx$

Let $u = x$ $\quad dv = e^{-x}dx$

$du = dx \quad V = -e^{-x}$

$\therefore V = 2\pi\left[-xe^{-x} + \int e^{-x}dx\right]_0^1$

$= 2\pi\left[-xe^{-x} - e^{-x}\right]_0^1$

$= 2\pi(-e^{-1} - e^{-1} - 0 + 1)$

$= 2\pi(1 - \frac{2}{e})$

$= 1.66$

c) $12 + L = L^2$

$L^2 - L - 12 = 0$

$(L-4)(L+3) = 0$

$\therefore L = -3, 4$

$\therefore L = 4$ as limit > 0

QUESTION 7

) Let roots be $\alpha, \alpha k, \alpha k^2$

$\therefore \quad \alpha(1+k+k^2) = -\frac{q}{p}$ —— ①

$\alpha^2 k(1+k+k^2) = \frac{r}{p}$ —— ②

$\alpha^3 k^3 = -\frac{s}{p}$ —— ③

) ÷ ① → $\alpha k = -\frac{r}{q}$ —— ④

ubst ④

to ③ ⇒ $-\frac{r^3}{q^3} = -\frac{s}{p}$

$pr^3 = q^3 s$

) $(z-i)(z+i) = z^2+1$

$$z^2+1 \,\overline{\smash{)}\, z^4 + 2z^3 - 2z^2 + 2z - 3}^{\;z^2+2z-3}$$

Now $z^2 + 2z - 3 = (z+3)(z-1)$

∴ other roots are $-i, 1, -3$

) $Q(x) = x^4 - 5x^3 + 4x^2 + 3x + 9$

$Q'(x) = 4x^3 - 15x^2 + 8x + 3$

$Q'(3) = 0$

$Q(3) = 0$

$x = 3$ double root

$(x-3)^2 = x^2 - 6x + 9$

$$x^2-6x+9 \,\overline{\smash{)}\, x^4 - 5x^3 + 4x^2 + 3x + 9}^{\;x^2+x+1}$$

$\therefore x = 3, \ -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$

) i) $0 \le x \le 4$

ii) $f'(x) = \frac{-1}{4\sqrt{x(2-\sqrt{x})}}$

< 0

R : $0 \le y \le \sqrt{2}$

iii) $y = \sqrt{2-\sqrt{x}}$

$x = (2-y^2)^2$

$= 4 - 4y^2 + y^4$

$A = \int_0^{\sqrt{2}} 4 - 4y^2 + y^4 \, dy$

$= \left[4y + \frac{4y^3}{3} + \frac{y^5}{5}\right]_0^{\sqrt{2}}$

$= \frac{32\sqrt{2}}{15}$.

✱ 8c) $x = mh + b$

when $h = 0$, $x = 20$

∴ $b = 20$

when $h = 10$, $x = 10$

∴ $10 = 10k + 20$

$k = -1$

∴ $x = 20 - h$

$A = \pi x \times \frac{x}{2}$

$= \frac{\pi}{2}(20-h)^2$

$V = \frac{\pi}{2}\int_0^{10} (20-h)^2 \, dh$

$= -\frac{\pi}{6}\left[(20-h)^3\right]_0^{10}$

$= 3665 \text{ m}^3$

QUESTION 8

a) i) $xy = 4$

$y = 4x^{-1}$

$\frac{dy}{dx} = \frac{-4}{x^2}$

when $x = 2p$

$\frac{dy}{dx} = \frac{-4}{4p^2}$

$= \frac{-1}{p^2}$

ii) $y - \frac{2}{p} = p^2(x - 2p)$

$yp - 2 = p^3 x - 2p^4$

$p^3 x - py = 2p^4 - 2$

$p^3 x - py = 2(p^4 - 1)$

iii) subst $(2q, \frac{2}{q})$ into normal

$p^3(2q) - \frac{2p}{q} = 2(p^4 - 1)$

$2p^3q^2 - 2p = 2p^4q - 2q$

$2p^3q^2 - 2p^4q = 2p - 2q$

$2p^3q(q - p) = 2(p - q)$

$p^3q = \frac{p-q}{q-p}$

$p^3q = -1$

OR $p^3x - py = 2(p^4 - 1)$ —— ①

$y = \frac{4}{x}$ —— ②

subst ⑤ into ① ⇒ $p^3x - \frac{4p}{x} = 2p^4 - 2$

$p^3x^2 - 4p = 2p^4x - 2x$

$p^3x^2 + x(2 - 2p^4) - 4p = 0$

now product of roots

$2p \times 2q = \frac{-4p}{p^3}$

$pq = -\frac{p}{\ }$.

$q = \frac{-1}{p^3}$

$p^3q = -1$

iv) to be normal at P & Q

$p^2 = q^2 \quad \therefore q = \pm p$

subst into $p^3q = -1$

∴ $p^4 = -1$ and $p^4 = 1$

∴ $p = \pm 1$

subst into eqn of normal

when $p = 1$, $p^3x - py = 2(p^4 - 1)$

$x - y = 0$

$y = x$

when $p = -1$, $p^3x - py = 2(p^4 - 1)$

$-x + y = 0$

$y = x$

∴ $y = x$

b) $y = \frac{5x}{3} + \frac{1}{3}$ —— ①

grad other diagonal $= -\frac{3}{5}$

∴ eqn diag ⇒ $y = -\frac{3x}{5} + b$

subst (3,11) into eqn of diag

$11 = -\frac{9}{5} + b$

$b = \frac{64}{5}$

∴ $y = -\frac{3x}{5} + \frac{64}{5}$ —— ②

solving ① + ② simultaneously

$\frac{5x}{3} + \frac{1}{3} = -\frac{3x}{5} + \frac{64}{5}$

$25x + 5 = -9x + 192$

$\left.\begin{aligned} x &= 5\tfrac{1}{2} \\ y &= 9\tfrac{1}{2} \end{aligned}\right\}$ middle of square

by symmetry (8,8) (7,12) (4,7)

✱ 8(c)