HORNSBY GIRLS' HIGH SCHOOL

2008
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using a black or blue pen
- Approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown for every question
- Begin each question on a fresh sheet of paper

Total marks (120)

- Attempt Questions 1 – 8
- All questions are of equal value

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Total Marks – 120

Attempt Questions 1 – 8
All Questions are of equal value

Begin each question on a NEW SHEET of paper, writing your student number and question number at the top of the page. Extra paper is available.

Question 1 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) Find $\int \frac{dx}{x^2 - 4x + 40}$ **2**

(b) Evaluate $\int_0^2 x^3 e^{x^2} dx$. **3**

(c) Find $\int \sin^3 x \, dx$ **2**

(d) Evaluate $\int_0^1 \frac{x}{\sqrt{4-x}} dx$ **3**

(e) (i) Find the real numbers a, b and c such that **3**

$$\frac{3x^2 + 2x + 11}{(x^2+3)(1-x)} \equiv \frac{ax+b}{x^2+3} + \frac{c}{1-x}.$$

(ii) Hence find $\int \frac{3x^2 + 2x + 11}{(x^2+3)(1-x)} dx$. **2**

Question 2 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) Given z is a complex number such that $z = 1 + i$

(i) Write z in mod-arg form **2**

(ii) Evaluate z^{12} **2**

(b) If $P(z) = z^4 - 30z^2 + 289$

(i) Show that $z = 4 + i$ is a zero of $P(z)$ **2**

(ii) Find all zeros of $P(z)$ over the complex field **5**

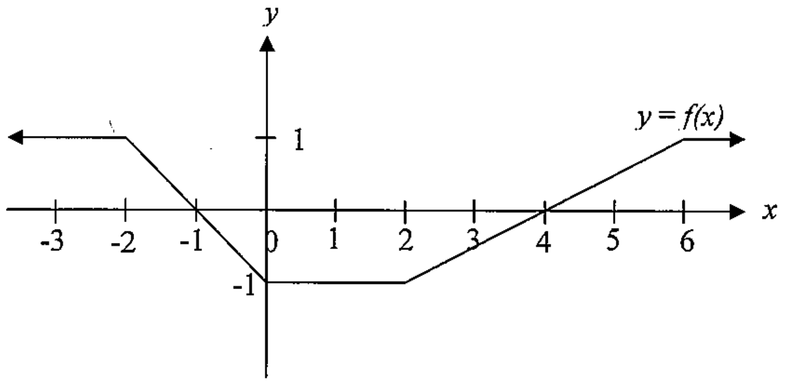
(c) $P(z)$ is a point on the argand diagram such that **4**

$$\arg \frac{z - i}{z + 2} = \frac{\pi}{2}$$

Draw and describe the locus of $P(z)$.

Question 3 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) The diagram below is a sketch of the function $y = f(x)$

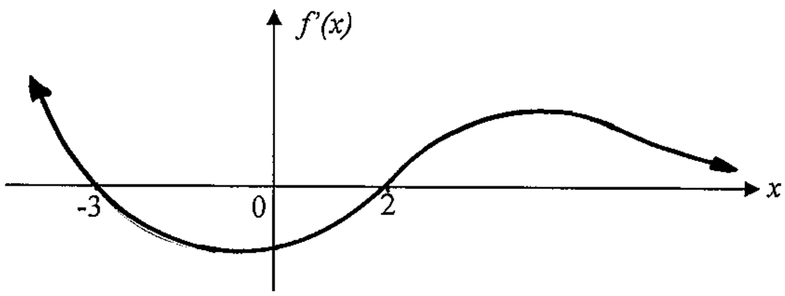


On separate diagrams sketch

(i) $y = |f(x)|$ **2**

(ii) $y = f(|x|)$ **1**

(b) The graph below represents the derivative $f'(x)$ of a certain function $f(x)$. Given that $f'(x) \to 0$ as $x \to \infty$, $f(0) = 0$ and $f(1) < 0$, sketch the graph of $f(x)$, noting the behaviour as $x \to \infty$. **3**



(c) (i) Sketch the curve $y = \dfrac{x^3 + 4}{x^2}$, showing any stationary points and asymptotic behaviour. **2**

(ii) Hence or otherwise, deduce the values of k, for which the equation $x^3 - kx^2 + 4 = 0$ may have one real root. **1**

(d) (i) If $x = a$ is a multiple root of the polynomial equation $P(x)$ such that $P(x) = 0$, prove that $P'(a) = 0$. **3**

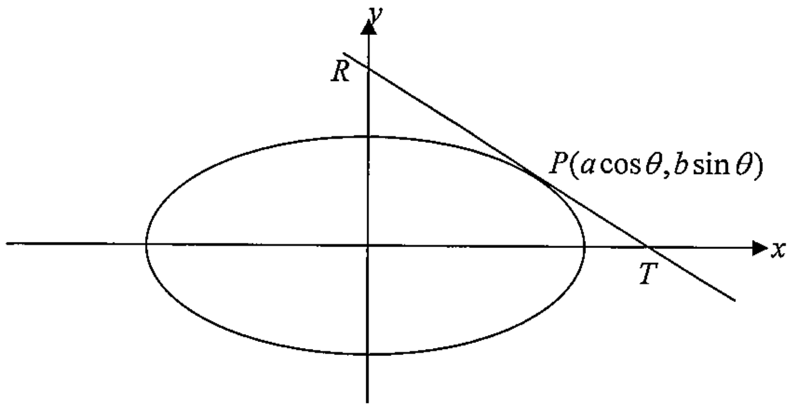
(ii) Find all roots of $P(x) = 16x^3 - 12x^2 + 1$ given that two of the roots are equal. **3**

Question 4 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) An ellipse has parametric equations $x = \sqrt{2}\cos\theta$ and $y = 3\sin\theta$. **2**

Find the Cartesian equation and the eccentricity of the ellipse.

(b)



The ellipse with equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ shown in the diagram above has a tangent at the point $P(a\cos\theta, b\sin\theta)$. The tangent cuts the x-axis at T and the y-axis at R.

(i) Show that the equation of the tangent at the point P is **2**

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1.$$

(ii) If T is the point of intersection between the tangent at point P and one of the directrices of the ellipse, show that $\cos\theta = e$. **3**

(iii) Hence find the angle that the focal chord through P makes with the x-axis. **1**

(iv) Using similar triangles or otherwise, show that $RP = e^2 RT$. **3**

(c) The area between the curve $y = \ln(x+1)$ and the x-axis, between $x = 0$ and $x = 1$ is rotated about the y-axis. **4**

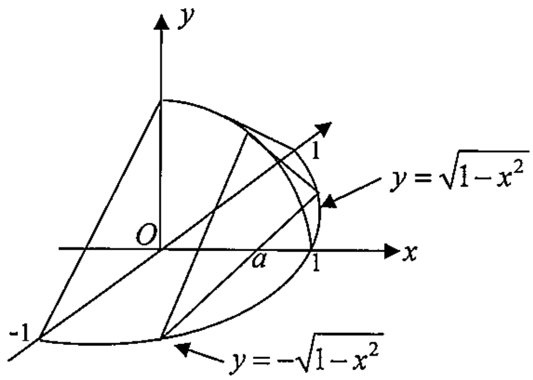
Find the volume of the solid of revolution formed using the method of cylindrical shells.

Question 5 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) (i) Write down the value of $\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx$. **1**

(ii) Explain why $\int_{-a}^{a} x\sqrt{a^2 - x^2}\, dx$ is equal to zero. **1**

(b)



The base of a solid is the semi-circular region in the x - y plane with the straight edge running from the point $(0,-1)$ to the point $(0,1)$ and the point $(1,0)$ on the curved edge of the semicircle.

Each cross-section perpendicular to the x-axis is an isosceles triangle with each of the two equal side lengths three quarters the length of the third side.

(i) Show that the area of the triangular cross-section at $x = a$ is $\frac{\sqrt{5}}{2}\left(1 - a^2\right)$. **2**

(ii) Hence find the volume of the solid. **2**

(c) The point $T\left(ct, \frac{c}{t}\right)$ lies on the hyperbola $xy = c^2$. The tangent at T meets the x-axis at P and the y-axis at Q. The normal at T meets the line $y=x$ at R.

(i) Prove that the tangent at T has equation $x + t^2 y = 2ct$. **2**

(ii) Find the coordinates of P and Q. **2**

(iii) Write down the equation of the normal at T. **1**

(iv) Show that the x coordinate of R is $x = \frac{c}{t}\left(t^2 + 1\right)$. **2**

(v) Prove that ΔPQR is isosceles. **2**

Question 6 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) The equation $x^3+2x-1=0$ has roots α, β, γ.

Find a polynomial equation in x whose roots are:

(i) $-\alpha, -\beta, -\gamma$ **1**

(ii) $\alpha^2, \beta^2, \gamma^2$ **2**

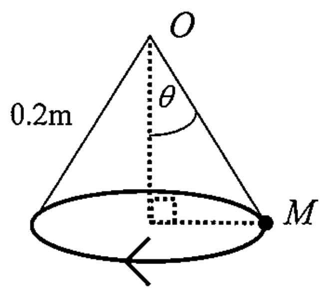
(iii) $\pm\alpha, \pm\beta, \pm\gamma$ **2**

(b) Find a and b if $(1+i)$ is a root of $x^2+(a+2i)x+5-ib=0$ **3**

(c) A body M, of mass 650g, is fixed to point O by a light wire 0.2m long.

The body rotates in a horizontal plane at 72 revolutions per minute.

Taking $g=10m/s^2$,



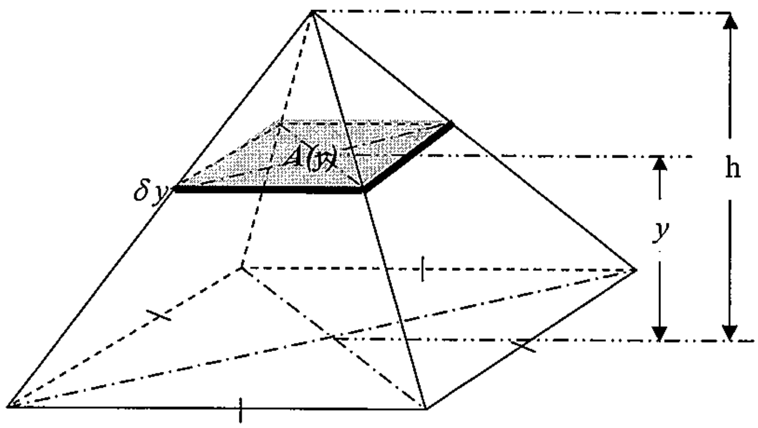
(i) Prove that $\tan\theta = \dfrac{72\pi^2 \sin\theta}{625}$. **3**

(ii) Find θ to the nearest minute. **2**

(iii) The mass of the body is to be doubled but the speed of rotation is to remain the same. What will happen to the value of θ? **2**

Question 7 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) The great pyramid of Cheops at Giza in Egypt is approximately 150 m high and its base is a square with an area of approximately 5 hectares.



(i) Show that the area of the cross section $A(y)$, at y is given by **1**

$$A(y) = (5\times10^4)\times\left(\frac{h-y}{h}\right)^2$$

(ii) Find the volume of the pyramid by using the slicing technique. **4**

(b) A particle of mass 1 kg is projected vertically upwards with an initial velocity of 100 m/s in a medium in which the resistance force is equal to 0.01 times the square of the body's velocity, i.e. $0.01v^2$. Use $g = 10 m/s^2$.

(i) Show that the maximum height reached by the particle is $50\log_e 11$ metres. **4**

(ii) Will the downward velocity of the particle on its return to the point of projection be greater than, less than, or equal to 100m/s? Justify your answer. **2**

(iii) Calculate the actual downward velocity of the particle on its return to the point of projection. **4**

Question 8 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) Use the following identity to answer the following questions.

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}.$$

(i) Solve $x^4 + 4\sqrt{3}x^3 - 6x^2 - 4\sqrt{3}x + 1 = 0$. **3**

(ii) Hence show that

(1) $\tan\frac{\pi}{24} + \tan\frac{7\pi}{24} + 4\sqrt{3} = \tan\frac{5\pi}{24} + \tan\frac{11\pi}{24}$ **1**

(2) $\tan\frac{\pi}{24}\tan\frac{5\pi}{24} = \cot\frac{7\pi}{24}\cot\frac{11\pi}{24}$ **1**

(iii) Find the polynomial of least degree that has zeros **3**

$$\left(\cot\frac{\pi}{24}\right)^2, \left(\cot\frac{7\pi}{24}\right)^2, \left(\cot\frac{13\pi}{24}\right)^2, \left(\cot\frac{19\pi}{24}\right)^2.$$

(b) Let $I_n = \int_0^1 x\left(x^2-1\right)^n\,dx$ for $n = 0,1,2,\ldots$

(i) Use integration by parts to show that **3**

$$I_n = \frac{-n}{n+1}I_{n-1} \text{ for } n \geq 1.$$

(ii) Hence or otherwise show that **2**

$$I_n = \frac{(-1)^n}{2(n+1)} \text{ for } n \geq 0.$$

(iii) Explain why $I_{2n} > I_{2n+1}$ for $n \geq 0$ **1**

(iv) Explain whether or not $I_n > I_{n+2}$ for all $n \geq 0$. **1**

End of Examination

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$