Homebush Boys' High School

TRIAL HIGHER SCHOOL CERTIFICATE

2012

Mathematics
EXTENSION 2

Time allowed – Three hours plus 5 minutes reading time

Name: ______________________________

General Instructions:

- Write using black or blue pen
- All necessary working should be Shown for every question
- Board approved calculators may be used
- A multiple choice answer sheet is provided for the multiple choice section
- Begin each question, 11-16 in a new booklet
- Attempt all questions
- All questions 11-16 are of equal value
- Total marks **(100)**

Section I

Questions 1 – 10 (1mark for each question)

Read each question and choose an answer A, B, C or D.
Record your answer on the Answer Sheet provided.
Allow about 15 minutes for this section

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1. The roots of the polynomial $x^3 - 5x + 4 = 0$ are α, β, and γ.
   What is the value of $\alpha^2 + \beta^2 + \gamma^2$?

   A) 25 B) 10 C) 5 D) – 5

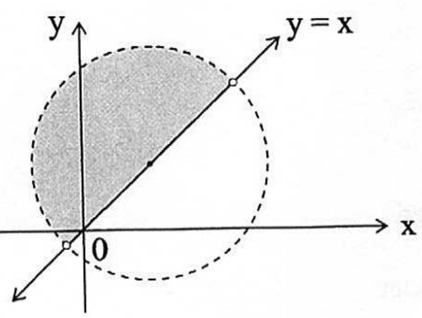
2. Evaluate $\int_0^{\frac{\pi}{4}} x\sec^2 x\,dx$.

   A) $\frac{\pi}{4} - \frac{1}{2}\ln 2$ B) $\frac{\pi}{4} + \frac{1}{2}\ln 2$ C) $\frac{\pi}{4} + 2\ln\sqrt{2}$ D) $\frac{7}{3}$

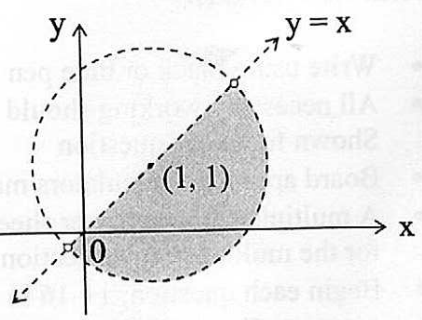
3. The complex number z satisfies the inequations
   $|z + 1 + i| < 2$ and $\text{Im}(z) \geq \text{Re}(z)$.

   Which of the following shows the shaded region in the Argand diagram that satisfies these inequations?

A)



B)



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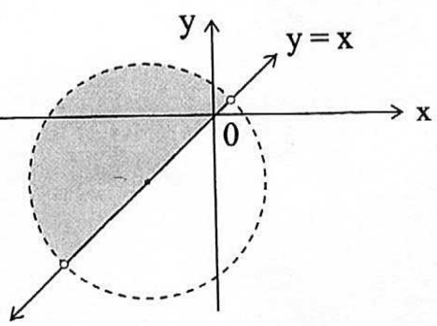
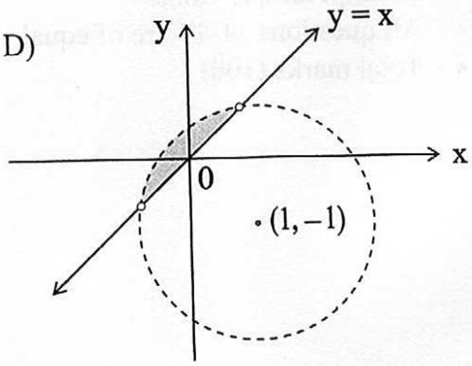


D)



4. The foci and the directrices of the ellipse with equation $4x^2 + y^2 = 4$ are

A) $(\pm\sqrt{3}, 0)$ and $x = \pm\frac{4\sqrt{3}}{3}$ B) $(0, \pm\sqrt{3})$ and $y = \pm\frac{4\sqrt{3}}{3}$

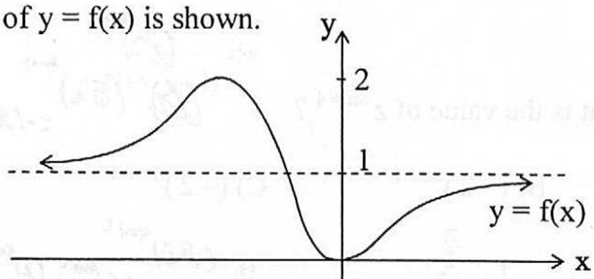
C) $(0, \pm\sqrt{3})$ and $x = \pm\frac{4\sqrt{3}}{3}$ D) $(\pm\sqrt{3}, 0)$ and $y = \pm\frac{4\sqrt{3}}{3}$

5. $\int(\tan^3 2x + \tan 2x)\,dx$ can be expressed as

A) $\frac{1}{4}\tan^4 2x + \frac{1}{2}\sec^2 2x + c$ B) $2\tan^2 2x + c$

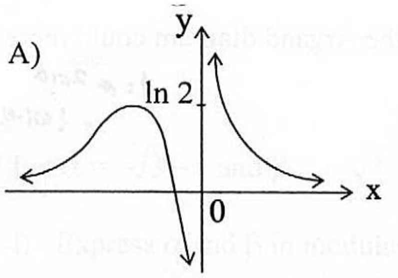
C) $\frac{1}{4}\tan^2 2x + c$ D) $\frac{1}{2}\tan^2 2x + c$

6. The graph of $y = f(x)$ is shown.



Which of these graphs could be the graph of $y = \ln(f(x))$

A)



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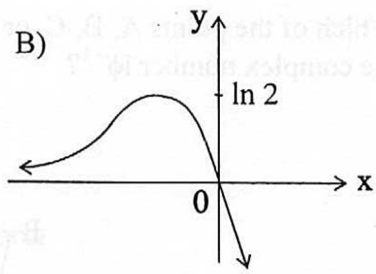

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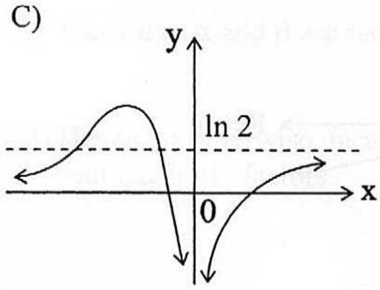
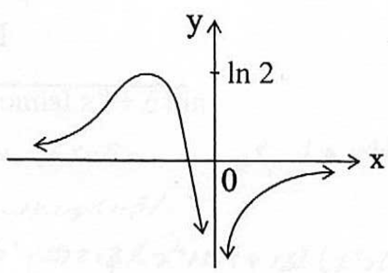


D)



7. $P(z)$ is a polynomial of degree 5 with $P(z) = P(\overline{z})$.

Which of the following statements must be false?

A) All the roots can be real.

B) Only four of the roots can be real.

C) Only three of the roots can be real.

D) Only one of the roots can be real.

8. The complex number z lies on the curve $|z - (1 + i)| = 1$.

What is the minimum value of $|z|$?

A) 1

B) $\sqrt{2}$

C) $\sqrt{2} - 1$

D) $\sqrt{2} + 1$

9. Let $z^2 = \sqrt{2}\,i$, what is the value of z^{8n+4}?

A) -2^{2n+1}

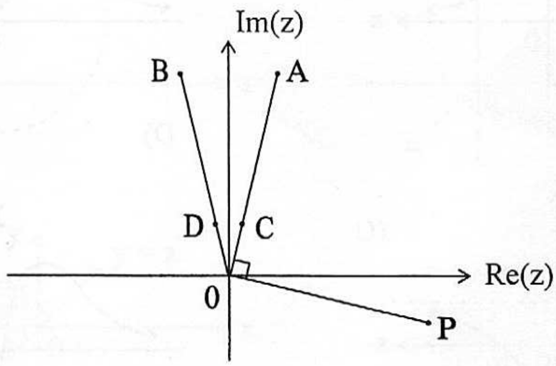
B) $(-2)^{8n+2}$

C) $(-2)^{6n+4}$

D) 2^{2n+1}

10. The point P represents the complex number ϕ where $\phi\overline{\phi} = 4$.

Which of the points A, B, C, or D shown in the Argand diagram could represent the complex number $i\phi^{-1}$?



Section II

Question 11 – 16 (15 marks each)

Allow about 2 hours 45 minutes for this section

Question 11 **MARKS**

a) i) Find the numbers a and b such that

$$\frac{3x^2+x}{(x+1)(x^2+1)} \equiv \frac{a}{x+1} + \frac{2x+b}{x^2+1}$$ 2

ii) Find $\int \frac{3x^2+x}{(x+1)(x^2+1)}\,dx$ 2

b) Use the substitution $t = \tan\frac{\theta}{2}$ to evaluate $\int_0^{\frac{\pi}{3}} \frac{1}{1-\sin\theta}\,d\theta$ 4

c) Let $\alpha = \sqrt{3} - i$ and $\beta = -\sqrt{3} + i$

i) Express α and β in modulus argument form. 2

ii) Show that α and β are roots of $z^6 + 64 = 0$ 2

iii) Hence, or otherwise, factorise the polynomial $z^6 + 64$ in real quadratic factors. 3

Question 12 **MARKS**

a) i) Find the two pairs of integers a and b such that $(a + ib)^2 = -9 + 40i$ 2

ii) Hence, solve the quadratic equation $z^2 + 5iz - 4 - 10i = 0$ 2

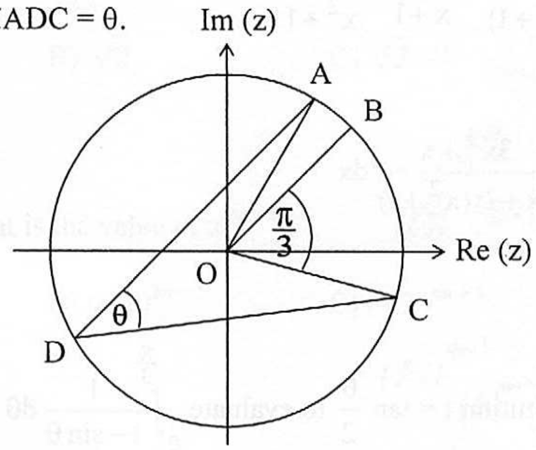
b) The polynomial $x^3 + ax^2 + bx + c = 0$ has roots α, β and $2(\alpha + \beta)$.

i) Show that $\alpha + \beta = -\frac{a}{3}$ 1

ii) Show that $4a^3 + 27c = 18ab$ 3

c) The points A, B, C and D representing respectively the complex numbers $\sqrt{2} + \sqrt{6}i$, $2 + 2i$, ω and ϕ lie on a circle with centre O.

$\angle BOC = \frac{\pi}{3}$ and $\angle ADC = \theta$.

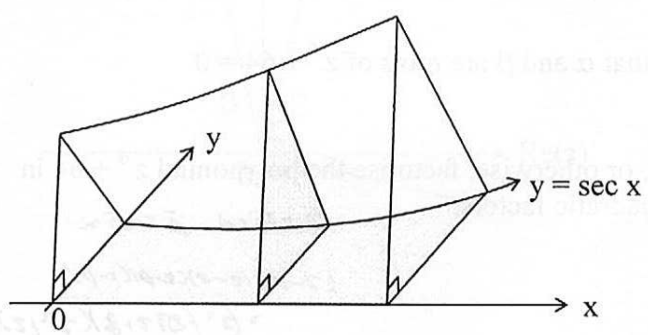


i) Find ω in the form x + iy, where x and y are real numbers. 2

ii) Find the value of θ. 2

d) The base of a solid is the region bounded the curve y = sec x, the x axis between x = 0 and $x = \frac{\pi}{4}$.

Vertical cross sections of the solid taken parallel to the y axis are in the shape of a right angled triangle with its height equal to the square of its base.

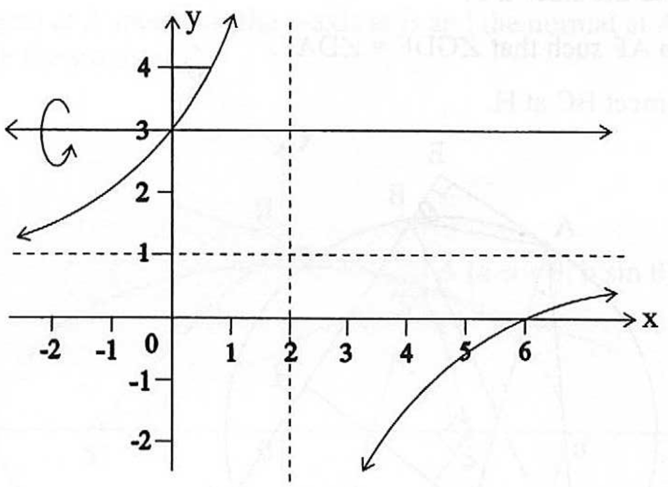


Find the volume of the solid. 3

Question 13 **MARKS**

a) Find the equation of the normal to the curve $2x^2 + xy + y^2 = 8$ at the point $(-1, 3)$. 3

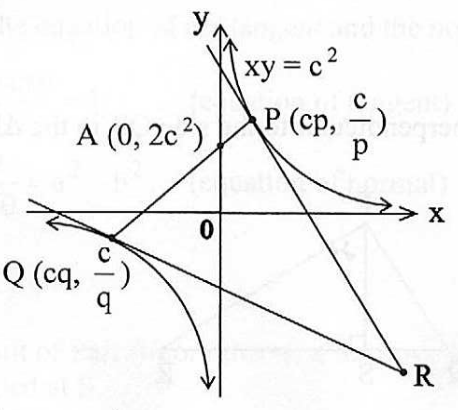
b) The diagram shows the graph of the hyperbola $f(x) = \dfrac{x-6}{x-2}$. 4



The area bounded by the curve $y = f(x)$, the y axis and the line $y = 4$ is rotated about the line $y = 3$ to form a solid.

Use the method of cylindrical shells to find the volume of the solid.

c) The tangents to the curve $xy = c^2$ at the points $P\left(cp, \dfrac{c}{p}\right)$ and $Q\left(cq, \dfrac{c}{q}\right)$ intersect at R. The chord PQ passes through the point $A(0, 2c^2)$.



i) Show that $2cpq = p + q$. 2

ii) Show that the equation of the tangent at P is $x + p^2y = 2cp$. 2

iii) Find the locus of R as P and Q vary. 4

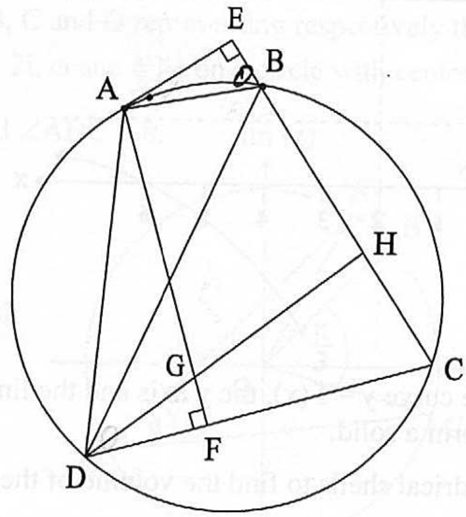
Question 14 **MARKS**

a) In the diagram ABCD is a cyclic quadrilateral.

AE is a perpendicular to the side CB produced and AF is the perpendicular to the sides DC.

G is a point on AF such that $\angle GDF = \angle DAF$.

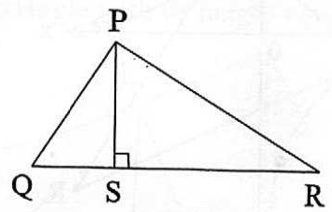
DG produced meet BC at H.



i) Show that ΔAEB is similar to ΔAFD. 2

ii) Show that the quadrilateral ABHG is cyclic. 3

b) In the diagram, PS is a perpendicular to the side QR in the ΔPQR.

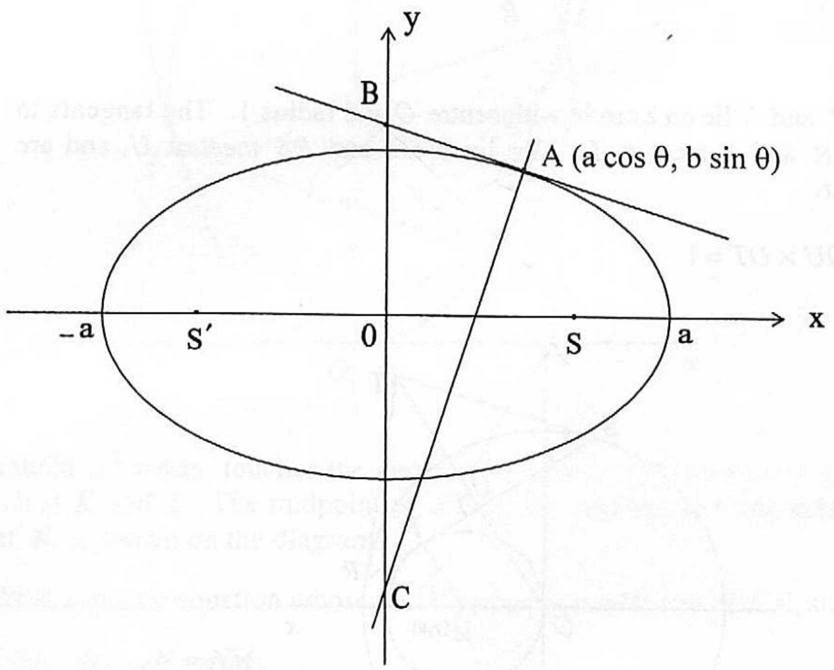


Given that $\frac{PS}{SR} = \frac{QS}{PS}$, show that ΔQPR is right angled at P. 2

Question 14 (Continued) **MARKS**

c) The point A $(a\cos\theta, b\sin\theta)$ lies on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ where S and S′ are the foci.

The tangent at A intersects the y-axis at B and the normal at A intersects the y axis at C.



i) Show that the equation of the tangent and the normal at A are 3

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1. \qquad \text{(equation of tangent)}$$

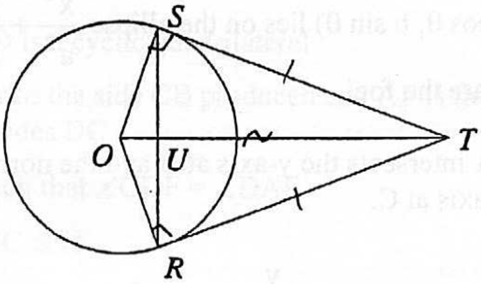
$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2. \qquad \text{(equation of normal)}$$

ii) Use the result of Part (b) or otherwise to prove that ΔBSC is right angled at S. 3

iii) Show that ∠ASB = ∠AS′B. 2

Question 15 **MARKS**

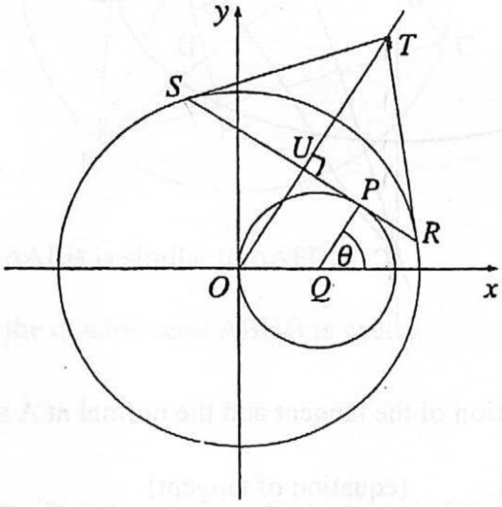
a) 2



The points R and S lie on a circle with centre O and radius 1. The tangents to the circle at R and S meet at T. The lines OT and RS meet at U, and are perpendicular.

Show that $OU \times OT = 1$.

b) 7



The circle $(x-r)^2 + y^2 = r^2$, with centre $Q(r,0)$ and radius r, lies inside the circle $x^2 + y^2 = 1$, with centre O and radius 1. The point $P(r + r\cos\theta, r\sin\theta)$ lies on the inner circle, and P and O do not coincide. The tangent to the inner circle at P meets the outer circle at R and S, and the tangents to the outer circle at R and S meet at T. The lines OT and RS meet at U, and are perpendicular.

(i) Show that OT is parallel to QP.

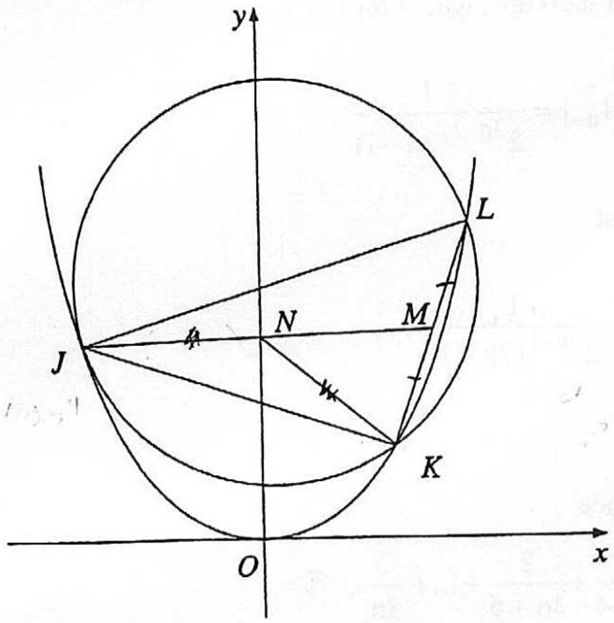
(ii) Show that the equation of RS is $x\cos\theta + y\sin\theta = r(1+\cos\theta)$.

(iii) Find the length of OU.

(iv) By using the result of part (a), show that T lies on the curve $r^2y^2 + 2rx = 1$.

Question 15 (continued) **MARKS**

c) 6



The parabola $x^2 = 4ay$ touches the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at J, and cuts it at K and L. The midpoint of KL is M, and the line JM cuts the y axis at N, as shown on the diagram.

(i) Find a quartic equation whose roots are the x coordinates of J, K, and L.

(ii) Show that $JN = NM$.

(iii) Hence show that the area of ΔJKN is one-quarter of the area of ΔJKL.

Question 16 **MARKS**

a) Let $I_n = \int_0^{\frac{\pi}{6}} \sin^{2n}\theta \sec\theta \, d\theta$, where for $n \geq 1$.

i) Show that $I_n - I_{n-1} = \dfrac{-1}{2^{2n-1}(2n-1)}$. 3

ii) Hence, show that 3

$$I_n = \frac{1}{2}\ln 3 - \sum_{k=1}^{n} \frac{1}{2^{2k-1}(2k-1)} .$$

b) Consider the sequence

$$S_k = \frac{2}{3n+2} + \frac{2}{3n+4} + \frac{2}{3n+6} + \ldots + \frac{2}{5n} ,$$

where n is a positive integer.

i) Show that $0 < S_k < \frac{2}{3}$. 2

ii) Given that $t < \theta < t+2$, where t is a positive integer and θ is a real number. 3

Show that $\dfrac{2}{t+2} < \displaystyle\int_t^{t+2} \frac{d\theta}{\theta} < \frac{2}{t}$.

iii) Show that $\ln\left(\dfrac{5n+2}{3n+2}\right) < S_k < \ln\dfrac{5}{3}$. 4

Hence, find the limit of S_k when $n \to +\infty$.