Trial Higher School Certificate 2011
Extension 2

Time: 3 hours **Total 120 marks**

Question 1 (15 marks) (begin on a new page)

Marks

a) Find $\int \frac{x}{\sqrt{16-x^2}}\,dx$. **2**

b) By completing the square, find

$$\int \frac{8}{x^2+4x+13}\,dx.$$ **2**

c) i) Find real numbers a, b, c, such that

$$\frac{7x+4}{\left(x^2+1\right)\left(x+2\right)} = \frac{ax+b}{x^2+1} + \frac{c}{x+2} \quad .$$ **3**

ii) Hence, find

$$\int \frac{7x+4}{\left(x^2+1\right)\left(x+2\right)}\,dx \quad .$$ **2**

d) Use integration by parts to find

$$\int x^2 \log_e x\,dx \quad .$$ **3**

e) Use the substitution $u = \cos x$, to find

$$\int \cos^2 x \sin^7 x\,dx.$$ **3**

Question 2 (15 marks) (begin on a new page) **Marks**

a) Let $z = 2 + i,\ w = 1 - i$. Find, in the form $x + iy$,

i) $3z + iw$, **1**

ii) $z\overline{w}$, **1**

iii) $\frac{5}{z}$. **1**

b) Let $\alpha = -\sqrt{3} + i$.

i) Express α in modulus - argument form. **2**

ii) Express α^4 in modulus - argument form. **1**

iii) Hence express α^4 in form $x + iy$. **1**

c) Sketch the region in the complex plane where the inequalities

$|z - i| \leq 2$ and $0 \leq \arg(z - 1) \leq \frac{3\pi}{4}$ hold simultaneously. **3**

d) Let $z_1 = 4 + i$ and $z_2 = 1 + 2i$. Let points *A*, *B*, *C* represent the complex numbers z_1, z_2, $z_1 - z_2$, respectively, on the complex plane.

i) On a diagram, show the points *A*, *B*, *C*. Indicate any geometrical relationships on your diagram. **1**

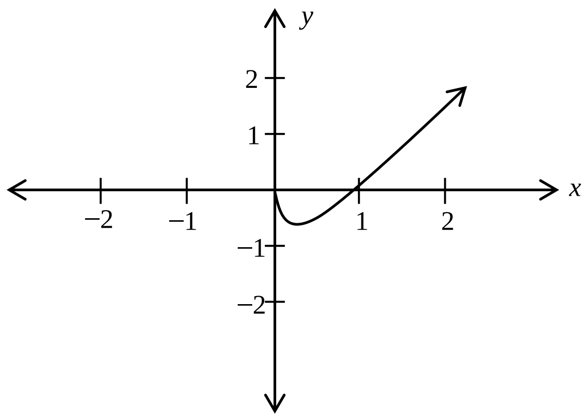
ii) The point *A* is rotated through 90° in the anticlockwise direction about *B* to the point *D*. Write down the complex number represented by *D*. **1**

e) On an Argand diagram, sketch and describe geometrically the locus of *z* such that

$|z| = |z - 4|$. **3**

Question 3 (15 marks) (begin on a new page) **Marks**

a) The diagram below shows the graph of the function $y = f(x)$.



Draw separate, one-third page sketches of the graphs of the following:

i) $y = -f(x)$, **1**

ii) $y = |f(x)|$, **1**

iii) $y = \dfrac{1}{f(x)}$, **2**

iv) $y = [f(x)]^2$. **2**

b) i) On the same set of axes, sketch the graphs of $y = \log_e x$ and $y = \dfrac{2}{x}$. **1**

ii) Hence, on a separate diagram, sketch the graph of $y = \dfrac{2\log_e x}{x}$. **4**

Indicate on your graph any asymptotes and the co-ordinates of any stationary points.

c) Sketch the graph of $y^2 = (x-2)(x-3)$, including any asymptotes. **4**

Question 4 (15 marks) (begin on a new page) **Marks**

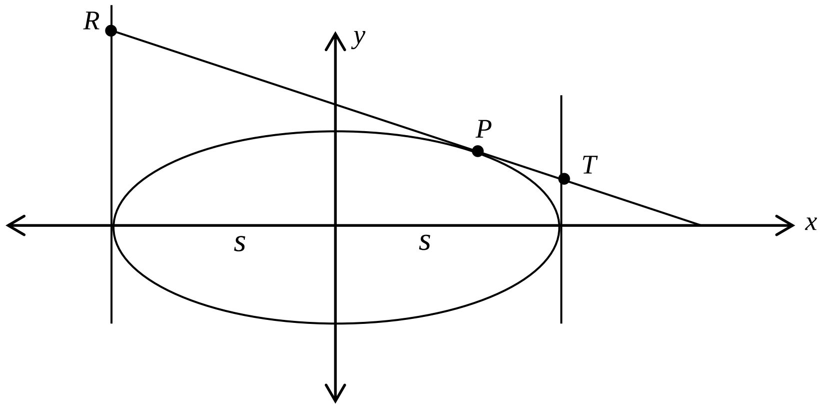
a) Consider the hyperbola with the equation $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

i) What is the eccentricity of the hyperbola? **1**

ii) Find the co-ordinates of the foci and x intercepts of the hyperbola. **2**

iii) Find the equations of the directrices and the asymptotes for the hyperbola. **2**

b)



The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a > b > 0, as shown in the diagram above. The points S and S' are the foci. The tangent at *P* meets the tangents at the ends of the major axis at *R* and *T*.

i) Show that the equation of the tangent at *P* is given by
$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1.$$ **2**

ii) Show that *RT* subtends a right angle at *S*. **3**

iii) Show that the points R, T, S, S' are concyclic. **1**

c) Let $Q(x_0, y_0)$ be an external point of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. (That is $\frac{x_0^{\ 2}}{a^2} + \frac{y_0^{\ 2}}{b^2} > 1$).
Show that the equation of the chord of contact of the tangents from the point Q to the ellipse, is given by the equation $\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$. **4**

Question 5 (15 marks) (begin on a new page) **Marks**

a) When $P(x) = x^4 + ax^2 + bx$ is divided by $x^2 + 1$, the remainder is $x + 2$.
Find the values of a and b, given that these values are real. **3**

b) The graph of $y = 2x^3 - 3x^2 - 12x + k$ has turning points at $x = 2$ and $x = -1$.
Find the values of k such that the equation $y = 2x^3 - 3x^2 - 12x + k = 0$ has three real and distinct roots. **2**

c) A nine-member fund raising committee consists of four students, three teachers and two parents. The committee meets around a circular table, such that all the students sit together as a group, all the teachers sit together as another group, but no teacher sits next to a student.

i) How many different arrangements are possible for the members of the committee to sit around the table. **2**

ii) One student and one parent are related. Given that all arrangements in b i) are equally likely, what is the probability that these two members sit next to each other? **2**

d) i) Write down the three relations which hold between roots α, β, γ of the equation $ax^3 + bx^2 + cx + d = 0$, $a \neq 0$ and the coefficients a, b, c, d. **1**

ii) Consider the equation $36x^3 - 12x^2 - 11x + 2 = 0$.
You are given that the roots α, β, γ of this equation satisfy $\alpha = \beta + \gamma$
Use part i) to find α. **2**

iii) Suppose the equation $x^3 + px^2 + qx + r = 0$ has roots α, β, γ, which satisfy $\alpha = \beta + \gamma$.
Show that $p^3 - 4pq + 8r = 0$. **3**

Question 6 (15 marks) (begin on a new page) **Marks**

a)

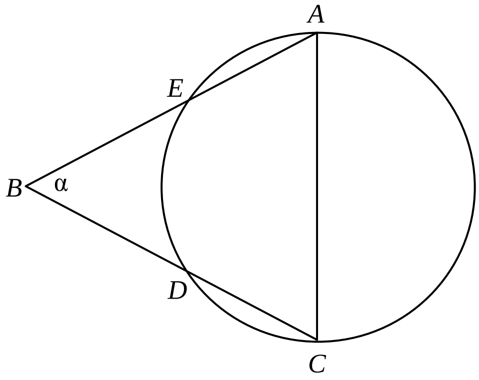


Diagram not to scale

In the diagram above, AC is a diameter of a circle with the point B outside the circle. The intervals BC and BA meet the circle in the points D and E respectively.

Also, $AC = BC$. Let $BA = x$, $BC = y$ and $\angle ABC = \alpha$.

i) Show that $\cos\alpha = \dfrac{x}{2y}$. **2**

ii) Find the length DC in terms of x and y. **4**

b) Let $I_n = \int_0^{\frac{\pi}{2}} \cos^n t \; dt$, where $n \geq 0$ is an integer.

i) Show that $I_n = \left(\dfrac{n-1}{n}\right) I_{n-2}$ for $n \geq 2$. **2**

ii) Hence find the exact value of I_4. **2**

Question 6 continues on the next page.

6c) i) Let $k > 0$ be a real number. **Marks**

If $\dfrac{(k+1)^{k-1}}{k^k} < 1$, show that $(k+1)^{k+1} > \dfrac{(k+1)^{2k}}{k^k}$. **2**

ii) Prove, by mathematical induction, for all integers $n \geq 2$ that

$n^n > (n+1)^{n-1}$. **3**

Question 7 (15 marks) (begin on a new page)

a) Let $I_n = \displaystyle\int \frac{dx}{(x^2+1)^n}$ where $n \geq 1$, is an integer.

i) Show that, for $n \geq 2$, $I_n = \dfrac{1}{2(n-1)}\left[\dfrac{x}{(x^2+1)^{n-1}} + (2n-3)I_{n-1}\right]$. **4**

ii) Hence evaluate

$\displaystyle\int_0^1 \frac{dx}{(x^2+1)^2}$. **2**

Question 7 continues on the next page.

7b)

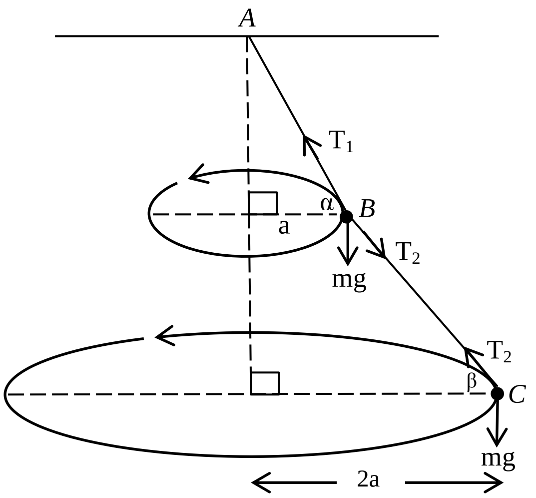


Diagram not to scale

A light, inextensible string ABC where $AB = \dfrac{5a}{3}$ and is inclined at an angle of α to the horizontal, while $BC = \dfrac{5a}{4}$ and is inclined at an angle β to the horizontal.
At B is attached a particle of mass $7m$ and at C is attached a particle of m.
The end A is attached to a fixed point and the whole system rotates steadily with uniform angular velocity about the vertical through A in such a way that B and C describe horizontal circles of radii a and $2a$ respectively.
The tensions in the strings AB and BC are T_1 and T_2 respectively. The strings remain taught.
The acceleration due to gravity is g.

i) Show that $T_2 = \dfrac{5}{3}mg$. **3**

ii) Find T_1. **2**

iii) If v_1 is the speed of B and v_2 is the speed of C, find the value of $\dfrac{v_1}{v_2}$. **4**

Question 8 (15 marks) (begin on a new page) **Marks**

a) The region between the curve $y = \sin x$ and the line $y = 1$, from $x = 0$ to $x = \dfrac{\pi}{2}$, is rotated around the line $y = 1$.
Using a slicing technique, find the exact volume formed. **4**

b) i) Differentiate with respect to x, the function, $h(x)$ given by

$$h(x) = \frac{\log_e x}{x} \quad \text{for } x > 0.$$ **1**

ii) Given that the only stationary point of $y = h(x)$ is a maximum turning point, deduce, without calculating any numerical values, that $e^{\pi} > \pi^{e}$.
$\left[\text{you may assume that } \pi > e\right]$ **2**

c) The diagram shows a boat showroom built on level ground. The length of showroom is 100 m. At one end of the showroom, the shape is a square measuring 20 m by 20 m and at the other end an isosceles triangle of height 20 m and base 10 m. The trapezium *ABCD* is a cross section of the showroom taken parallel to the ends.

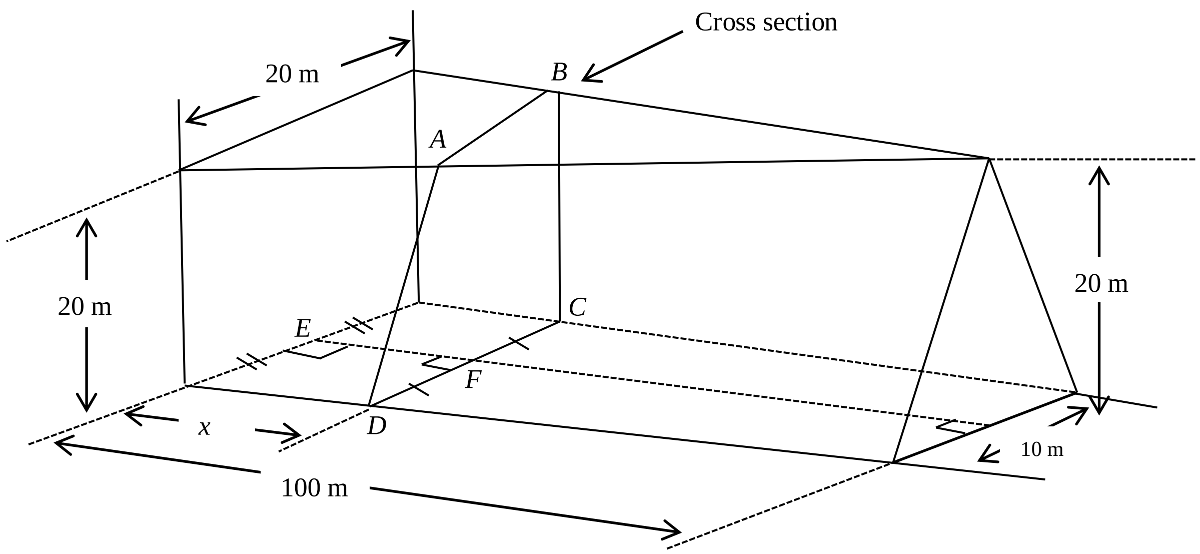


Diagram not to scale.

i) If *EF* is x metres in length, show that the length of *DC* is $\left(20 - \dfrac{x}{10}\right)$ metres. **2**

ii) By considering trapezoidal slices parallel to the ends of the showroom, find the volume enclosed by the showroom. **6**

END OF EXAMINATION