Total marks – 120
Attempt Question 1-8
All questions are of equal value
Answer each question in the appropriate writing booklet. Extra writing booklets are available.

Question 1 (15 marks) **Marks**

(a) Find

(i) $\displaystyle\int \frac{\cos\theta}{\sin^5\theta}\,d\theta$ **2**

(ii) $\displaystyle\int \frac{dx}{x^2+2x+2}$ **2**

(b) Use the substitution $t=\tan\frac{\theta}{2}$ to find $\displaystyle\int \frac{dx}{5+4\cos x+3\sin x}$ **3**

(c) Use the substitution $u=-x$ to evaluate $\displaystyle\int_{-1}^{1} \frac{dx}{e^x+1}$ **3**

(d) Evaluate the following definite integrals:

(i) $\displaystyle\int_0^1 \cos^{-1}x\,dx$ **2**

(ii) $\displaystyle\int_1^2 x(\ln x)^2\,dx$ **3**

Question 2 (15 marks) Start a new booklet

(a) If $z = 3 - 2i$ mark clearly on an Argand diagram the points represented by,

(i) $2z$ **1**

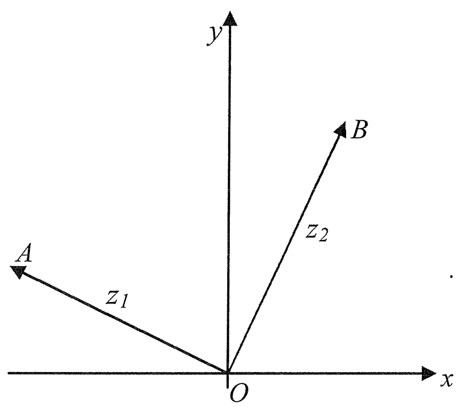
(ii) $-2iz$ **2**

(b) z is a complex number such that $\arg z = \frac{\pi}{3}$ and $|z| \le 2$.

(i) Sketch the locus of the point P representing z in the Argand diagram. **2**

(ii) Find the possible values of the principal argument of $z - 1$ for z on this locus. **2**

(c)



In the Argand diagram, vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represent the complex numbers $z_1 = 2\left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right)$ and $z_2 = 2\left(\cos\frac{7\pi}{15} + i\sin\frac{7\pi}{15}\right)$ respectively.

(i) Show that ΔOAB is equilateral **3**

(ii) Explain why $z_2 - z_1$ is equal to z_2 rotated by $\frac{\pi}{3}$ radians in a clockwise direction **2**

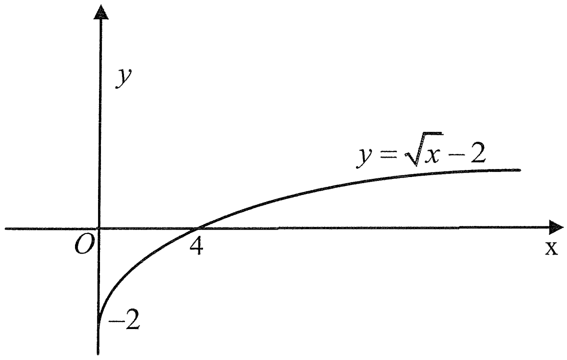
(iii) Express $z_2 - z_1$ in modulus-argument form. **3**

Question 3 (15 marks) Start a new booklet

(a) The polynomial $p(x) = x^4 - 2x^3 + 2x - 1$ has a root of multiplicity 3. Find this root and hence factorise $p(x)$ **3**

(b) Sketch the curve $y^2 = x^2(1 - x^2)$ clearly showing all relevant details. **6**

(c)



The diagram shows the graph of the function $f(x) = \sqrt{x} - 2$. On separate diagrams (each of half a page) sketch the following graphs, showing clearly any intercepts on the coordinate axes and the equations of any asymptotes:

(i) $y = |f(x)|$ **1**

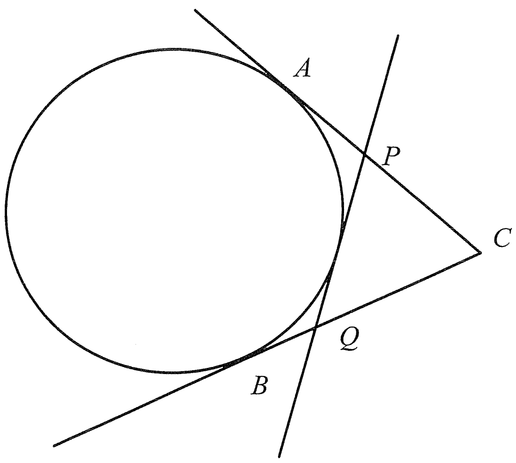
(ii) $y = [f(x)]^2$ **1**

(iii) $y = \dfrac{1}{f(x)}$ **2**

(iv) $y = \ln f(x)$ **2**

Question 4 (15 marks) Start a new booklet

(a) A and B are two points on a circle. Tangents at A and B meet at C. A third tangent cuts CA and CB in P and Q respectively, as shown in the diagram. Show that the perimeter of ΔCPQ is independent of PQ. **3**



(b) The polynomial $P(x)$ leaves a remainder of 9 when divided by $(x-2)$ and a remainder of 4 when divided by $(x-3)$. Find the remainder when $P(x)$ is divided by $(x-2)(x-3)$. **4**

(c) The polynomial $P(x) = x^4 + 3x^3 + 6x^2 + 12x + 8$ has one root $2i$. Find all the roots of $P(x)$. **4**

(d) If α, β are the roots of the equation $x^2 - px + q = 0$ and $S_n = \alpha^n + \beta^n$ where n is a positive integer, show that **4**

$$S_{n+2} - pS_{n+1} + qS_n = 0$$

Hence, or otherwise find S_3, S_4 in terms of p, q.

Question 5 (15 marks) Start a new booklet

(a) $P\left(3p, \frac{3}{p}\right)$ and $Q\left(3q, \frac{3}{q}\right)$ are points on different branches of the hyperbola $xy = 9$.

(i) Find the equation of the tangent at *P*. **2**

(ii) Find the point of intersection, *T*, of the tangents at *P* and *Q*. **2**

(iii) If the chord *PQ* passes through the point $(0, 2)$, find the locus of *T*. **3**

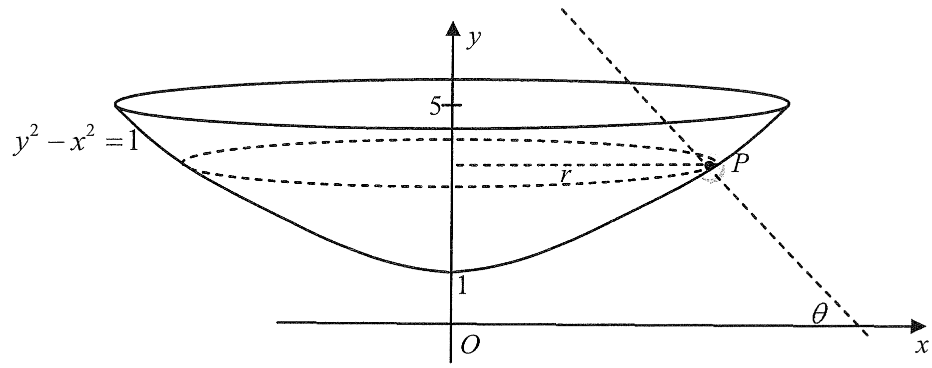
(b) The region bounded by the graphs of $y = x^2$ and $y = x + 2$ is revolved around the line $x = 3$. Derive the volume of the resulting solid as a definite integral. **Do not calculate the value of this integral**. **4**

(c) A solid has, as its base, the circular region in the *xy*-plane bounded by the graph of $x^2 + y^2 = a^2$, where $a > 0$. If every cross-section by a plane perpendicular to the *x*-axis is an equilateral triangle, with one side in the base, show that the volume of the solid is $\frac{4\sqrt{3}}{3}a^3$ units^3. **4**

Question 6 (15 marks) Start a new booklet

(a) A particle of mass m moves in a straight line away from a fixed point O in the line, such that at time t its displacement from O is x and its velocity is v. At time $t=0$, $x=1$ and $v=0$. Subsequently, the only force acting on the particle is one of magnitude $m\frac{k}{x^2}$, where k isa positive constant in a direction away from O. Show that v cannot exceed $\sqrt{(2k)}$. **4**

(b)



A bowl is formed by rotating the hyperbola $y^2 - x^2 = 1$ for $1 \le y \le 5$ about the y axis. A particle P of mass m moves around the inner surface of the bowl in a horizontal circle with constant angular velocity ω.

(i) Show that if the radius of the circle in which P moves is r, then the normal to the surface at P makes an angle θ with the horizontal as shown, where $\tan\theta = \frac{\sqrt{1+r^2}}{r}$. **4**

(ii) Draw a diagram showing the forces acting on P. **1**

(iii) By resolving these forces in the horizontal and vertical directions show that $r = \frac{\sqrt{g^2 - \omega^4}}{\omega^2}$ and the normal reaction $N = m\sqrt{2g^2 - \omega^4}$ **3**

(iv) Since P must be in contact with the surface of the bowl and the radius must be positive, prove $\sqrt{\frac{g}{5}} \le \omega \le \sqrt{g}$. **3**

Question 7 (15 marks) Start a new booklet

a) i) Expand $(2+i)(3+i)$ **1**

ii) By considering the arguments of each side, or otherwise show that **2**

$$\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)=\frac{\pi}{4}$$

iii) By first expanding $(p+q+i)(p^2+pq+1+iq)$ derive **4**

$$\tan^{-1}\left(\frac{1}{p+q}\right)+\tan^{-1}\left(\frac{q}{p^2+pq+1}\right)=\tan^{-1}\left(\frac{1}{p}\right)$$

b) i) Prove, by Mathematical Induction, that **3**

$\sin\theta = 2^n \cos\frac{\theta}{2}\cos\frac{\theta}{4}\ldots\cos\frac{\theta}{2^n}\sin\frac{\theta}{2^n}$ for any positive integer n

ii) Hence show that $\dfrac{\sin\theta}{\theta}=\cos\frac{\theta}{2}\cos\frac{\theta}{4}\ldots\cos\left(\frac{\theta}{2^n}\right)\dfrac{\sin\left(\frac{\theta}{2^n}\right)}{\left(\frac{\theta}{2^n}\right)}$ **1**

iii) Hence find $\lim\limits_{n\to\infty}\left(\dfrac{\sin\theta}{\theta}\right)$ **1**

iv) Using $\prod$ as the product of terms, show that when $\theta=\frac{\pi}{2}$ **1**

$$\prod_{k=2}^{\infty}\cos\left(\frac{\pi}{2^k}\right)=\frac{2}{\pi}$$

v) Hence show, by applying the half angle formula for $\cos\left(\frac{\theta}{2}\right)$ **2**

$$\frac{2}{\pi}=\frac{\sqrt{2}}{2}\times\frac{\sqrt{2+\sqrt{2}}}{2}\times\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\times\ldots$$

Question 8 (15 marks) Start a new booklet

(a) The ellipse $\mathscr{E}$: $\left(\frac{x}{5}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$ has foci $S(4,0)$ and $S'(-4,0)$.

(i) Sketch the ellipse $\mathscr{E}$ indicating its foci S, S' and its directrices. **1**

(ii) Find the tangent at $P(x_1, y_1)$ on the ellipse $\mathscr{E}$. **1**

(iii) The line joining $P(x_1, y_1)$ to $Q(x_2, y_2)$ passes through S. Show that $4(y_2 - y_1) = x_1 y_2 - x_2 y_1$. **2**

(iv) If it is also known that $Q(x_2, y_2)$ lies on $\mathscr{E}$ find the point of intersection of the tangents at P and Q **1**

(v) By using the result in (iii) show the tangents at P and Q intersect on the directrix corresponding to S. **2**

(b) $I_n = \int_1^e (1 - \ln x)^n \, dx$, $n = 1, 2, 3, \ldots$

(i) Show $I_n = -1 + nI_{n-1}$, $n = 1, 2, 3, \ldots$ **3**

(ii) Hence evaluate $\int_1^e (1 - \ln x)^3 \, dx$. **2**

(iii) Show that $\frac{I_n}{n!} = e - \sum_{r=0}^{n} \frac{1}{r!}$, $n = 1, 2, 3, \ldots$ **3**

End of Examination

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a \neq 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$