



2010
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 2

General Instructions

- Reading Time - 5 minutes.
- Working Time - 3 hours.
- Write using a blue or black pen.
- Board Approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown for every question.
- Begin each question in a new booklet

Total marks (120)

- Attempt Questions 1-8.
- All questions are of equal value.

2010 Extension 2 Trial Higher School Certificate

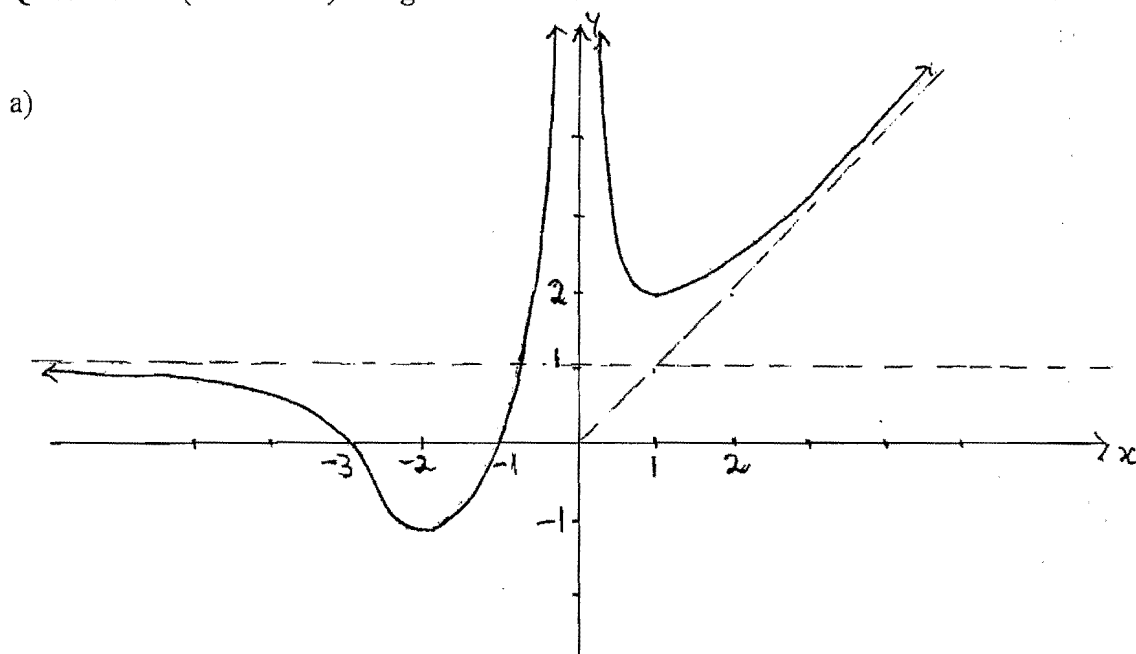
Question 1 (15 Marks)

Marks

- a) Find $\int \frac{x}{\sqrt{16-9x^2}} dx$ 1
- b) Find $\int_0^\pi x \sin x dx$ 3
- c) Use the substitution $u = \sqrt{x-1}$ to evaluate $\int_2^3 \frac{1+x}{\sqrt{x-1}} dx$ 3
- d) Use partial fractions to find $\int \frac{(2x^2 + 5x + 9)dx}{(x-1)(x^2 + 2x + 5)}$ 4
- e) Use the substitution $t = \tan \frac{\theta}{2}$ to show that $\int_0^{\frac{\pi}{3}} \frac{1}{1 + \sin \theta} d\theta = \sqrt{3} - 1$ 4

Question 2 (15 Marks) Begin a New Booklet

- a) Given $A = 3 + 4i$ and $B = 1 - i$, express the following in the form $x + iy$
- (i) AB 2
 - (ii) $\frac{A}{iB}$ 2
 - (iii) \sqrt{A} 3
- b) Let $\alpha = \sqrt{3} - i$
- (i) Find the exact value of $|\alpha|$ and $\arg \alpha$ 2
 - (ii) Find the exact value of α^5 in the form $a + ib$ where a and b are real. 2
- c) (i) On an Argand diagram, shade the region where
- $$|z - 1 - i| \leq \sqrt{2} \quad \text{and} \quad 0 \leq \arg z \leq \frac{\pi}{4} \quad \text{both hold.} \quad 2$$
- (ii) Find, in simplest exact form, the area of this shaded region. 2



The diagram shows the graph of $y = f(x)$.

Draw separate half page sketches of:

(i) $y = (f(x))^2$ 2

(ii) $y = \sqrt{f(x)}$ 2

(iii) $y^2 = f(x)$ 1

(iv) $y = \frac{1}{f(x)}$ 2

(v) $y = f'(x)$ 2

b) For the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$

(i) find the eccentricity 1

(ii) find the coordinates of the foci S and S' 1

(iii) find the equations of the directrices. 1

(iv) Sketch the curve showing the foci and directrices. 1

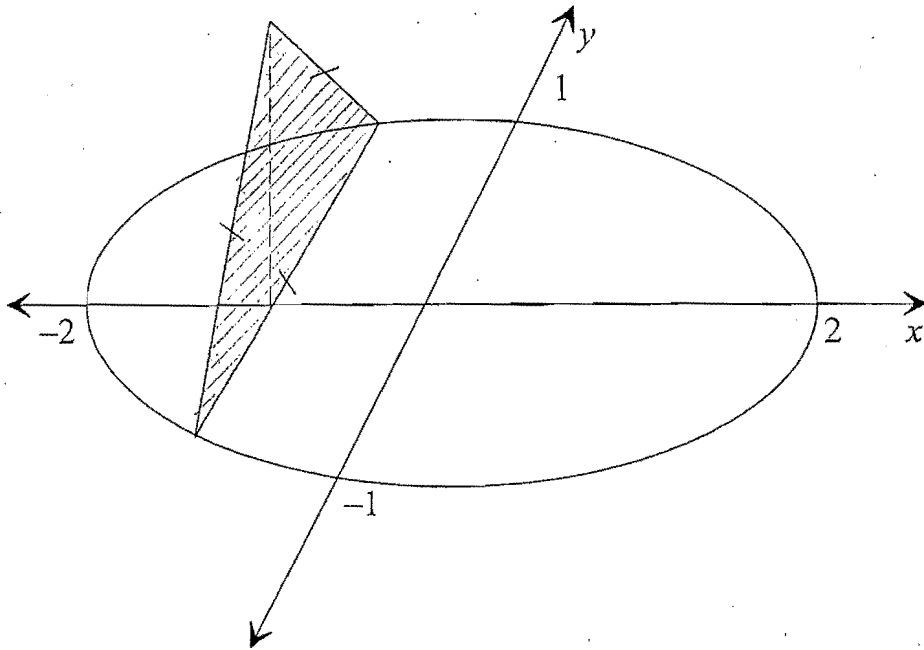
P is an arbitrary point on this ellipse.

(v) Prove that the sum of the distances SP and $S'P$ is independent of P . 2

Question 4 (15 Marks) Begin a New Booklet

Marks

- a) A solid shape has as its base an ellipse in the XY plane as shown below. Sections taken perpendicular to the X -axis are equilateral triangles. The major and minor axes of the ellipse are 4 metres and 2 metres respectively.

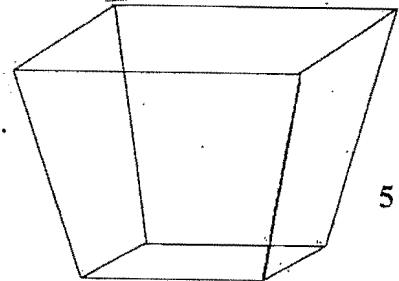


- i. Write down the equation of the ellipse. 1
- ii. Show that the area of the cross-section at $x = k$ is given by 2

$$A = \frac{\sqrt{3}}{4} (4 - k^2).$$

- iii. By using the technique of slicing, find the volume of the solid. 2

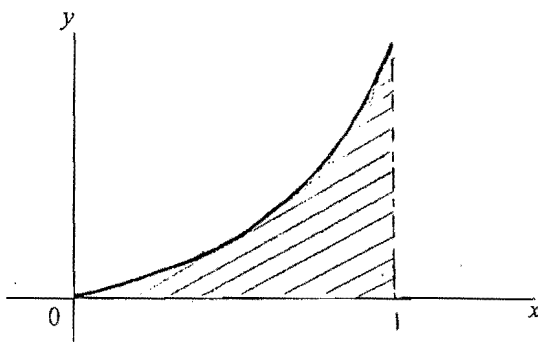
- b) A container has 6 plane faces. The top is a rectangle, 70cm by 50cm. The bottom is a rectangle, 30cm by 20cm, parallel to the top rectangle. The remaining faces are all trapezia. The perpendicular height is 20cm.



Use slices parallel to the base to find the capacity of the container in litres

- c) The area between the curve $y = e^{x^2} - 1$ and the x -axis, from $x = 0$ to $x = 1$ is rotated about the y -axis. 5

Sketch a typical cylindrical shell and use this method to find the volume formed.



Question 5 (15 Marks) Begin a New Booklet

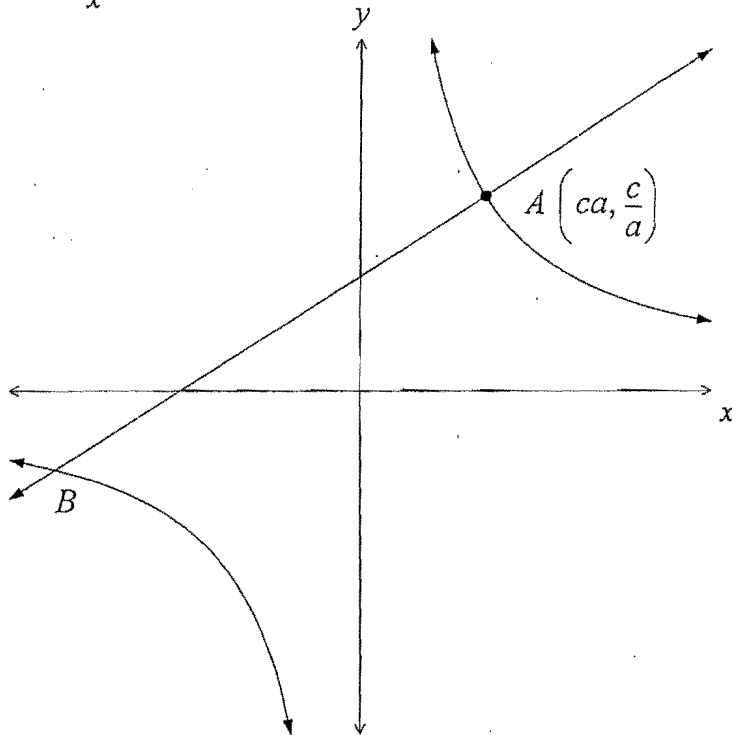
- a) The roots of the equation $x^3 - 3x^2 + 9 = 0$ are α , β and γ .
- (i) Find the polynomial equation with roots α^2 , β^2 and γ^2 . 2
- (ii) Find the value of $\alpha^2 + \beta^2 + \gamma^2$ and hence evaluate $\alpha^3 + \beta^3 + \gamma^3$. 2
- b) Given that the polynomial $P(x)$ has a double zero at $x = \alpha$, show that the polynomial $P'(x)$ will have a single zero at $x = \alpha$. 2
- c) When a polynomial $P(x)$ is divided by $(x-1)$ the remainder is 3 and when divided by $(x-2)$ the remainder is 5. Find the remainder when the polynomial is divided by $(x-1)(x-2)$. 3
- d) $P(x) = x^4 - 2x^3 + 3x^2 - 4x + 1$ and the equation $P(x) = 0$ has roots α , β , γ and δ .
- (i) Show that the equation $P(x) = 0$ has no integer roots. 1
- (ii) Show that $P(x) = 0$ has a real root between 0 and 1. 1
- (iii) Show that $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = -2$. 2
- (iv) Hence find the number of real roots of the equation $P(x) = 0$, giving reasons. 2

Question 6 (15 Marks) Begin a New Booklet

a) Show that $\frac{x^4 + x^2 + 1}{x^2} \geq 3$ for all x . (Hint: Consider $(x^2 - 1)^2$)

2

b)



The point $A \left(ca, \frac{c}{a} \right)$, where $a \neq \pm 1$ lies on the hyperbola $xy = c^2$. The normal through A meets the other branch of the curve at B .

i. Show that the equation of the normal through A is

2

$$y = a^2 x + \frac{c}{a} (1 - a^4)$$

ii. Hence if B has coordinates $\left(cb, \frac{c}{b} \right)$, show that $b = \frac{-1}{a^3}$.

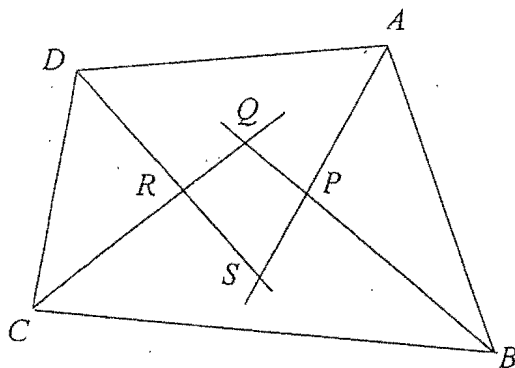
3

iii. If this hyperbola is rotated clockwise through 45° , show that the equation becomes

4

$$x^2 - y^2 = 2c^2.$$

c)



In the quadrilateral $ABCD$ shown above, APS , BPQ , CRQ and DRS are the bisectors of the vertex angles at A , B , C and D respectively.

(i) Show that $PQRS$ is a cyclic quadrilateral.

2

(ii) If $ABCD$ is a trapezium, deduce that one of the diagonals of $PQRS$ is a diameter of the circle through P , Q , R and S .

2

Question 7 (15 Marks) Begin a New Booklet

- a) A solid of unit mass is dropped under gravity from rest at a height of h metres.
Air resistance is proportional to the speed v of the mass, acceleration under gravity is g .

- (i) Using k as the constant of proportionality, show that the acceleration is given by 1

$$\frac{d^2x}{dt^2} = g - kv$$

- (ii) Show that the velocity v of the solid after t seconds is given by 3

$$v = \frac{g}{k}(1 - e^{-kt})$$

- (iii) Using the fact that $\frac{d^2x}{dt^2} = v \frac{dv}{dx}$ show that

$$x = \frac{g}{k^2} \left[\ln \frac{g}{g - kv} - \frac{kv}{g} \right] \quad 4$$

- b) Further from the surface of the Earth, we may neglect air resistance, but acceleration due to gravity is not constant. It varies inversely with the square of the distance from the centre of the Earth. That is

$$\frac{d^2x}{dt^2} = -\frac{k}{x^2} \quad \text{where } x \text{ is the distance measured from the } \underline{\text{centre}} \text{ of the Earth.}$$

- (i) If the acceleration due to gravity at the surface of the Earth is g , and the radius of the Earth is R , show that this constant is gR^2 1

- (ii) A rocket is fired vertically upwards from the surface of the Earth with an initial velocity of u . Neglecting air resistance, show that its velocity v is given by

$$v^2 = \frac{2gR^2}{x} + u^2 - 2gR \quad 2$$

- (iii) Find (in terms of u , g , and R) the greatest height possible if the rocket is to return to Earth. 2

- (iv) Hence find the initial velocity which must be exceeded for the rocket to escape the Earth's gravitational pull and never return. 2
(Use $g = 9.8 \text{ m/s}^2$ and $R = 6367 \text{ km}$)

Question 8 (15 Marks) Begin a New Booklet

- a) (i) Show that the recurrence (reduction) formula for

$$I_n = \int \tan^n x \, dx \quad \text{is} \quad I_n = \frac{1}{n-1} \tan^{n-1} x - I_{n-2} \quad 4$$

- (ii) Hence find the exact value of $\int_0^{\frac{\pi}{4}} \tan^3 x \, dx$ 3

- b) It is given that $z^5 = 1$ where $z \neq 1$

(i) Show that $z^2 + z + 1 + z^{-1} + z^{-2} = 0$ 1

(ii) Show that $z + z^{-1} = 2 \cos \frac{2k\pi}{5}$ for $k = 1, 2, 3, 4$ 2

- (iii) By letting $x = z + z^{-1}$ reduce the equation in (i) above to a quadratic equation in x . 3

(iv) Hence deduce that $\cos \frac{\pi}{5} \cdot \cos \frac{2\pi}{5} = \frac{1}{4}$ 2