2010

TRIAL

HIGHER SCHOOL CERTIFICATE

GIRRAWEEN HIGH SCHOOL

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board – approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

Attempt Questions 1 – 8

All questions are of equal value

Total marks-120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE piece of paper clearly marked Question 1, Question 2, etc. Each piece of paper must show your name.

Question 1 (15 Marks)

a) $\int \frac{dx}{\sqrt{4x^2-9}}$ **2**

b) $\int \frac{xdx}{\sqrt{4x^2-9}}$ **2**

c) (i) Find real numbers ***a, b*** and ***c*** such that $\frac{4x^2+7x+11}{(x+3)(x^2+4)} = \frac{a}{x+3} + \frac{bx+c}{x^2+4}$ **2**

(ii) Hence show $\int_0^2 \frac{4x^2+7x+11}{(x+3)(x^2+4)} = \ln\frac{50}{9} + \frac{\pi}{8}$ **2**

d) Use $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x}$ **2**

e) Find $\int \sqrt{x} \ln x dx$ **2**

f) Evaluate $\int_2^6 \frac{dx}{x\sqrt{2x-3}}$ **3**

Question 2 (15 marks)

a) (i) Find real numbers $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\sqrt{9-40i} = a+ib$ 2

(ii) Hence find the solutions to $z^2 - 3z + 10i = 0$ 2

b) (i) Sketch the graph of $|z-4i| = 2$ 2

(ii) Hence find the greatest and least values of $arg\, z$ 2

c) If $\omega = \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}$ where k is an integer and $\omega \neq 1$

(i) Show that $\omega^n + \omega^{-n} = 2\cos\frac{2nk\pi}{5}$ 2

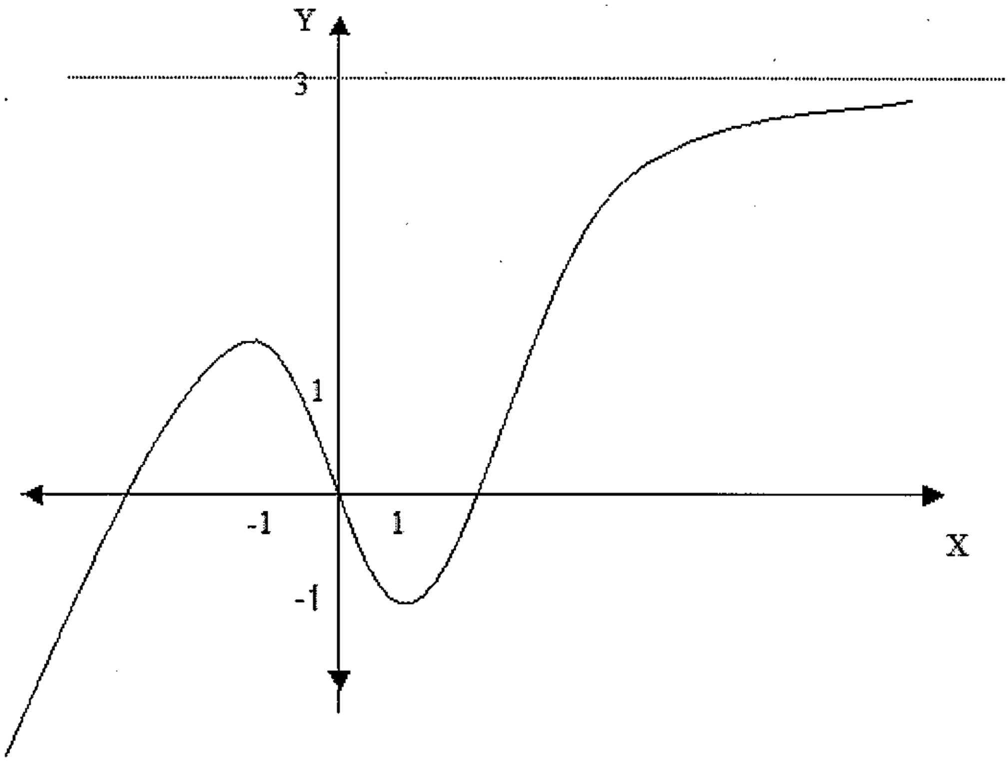
(ii) Show that $\omega^5 = 1$ 1

(iii) Show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$ 1

(iv) Hence or otherwise show that $(\omega + \omega^{-1})^2 + (\omega^2 + \omega^{-2})^2 = 3$ 2

(v) Deduce that $(\cos\frac{2k\pi}{5})^2 + (\cos\frac{4k\pi}{5})^2 = \frac{3}{4}$ 1

Question 3 (15 Marks)



a) The graph of $y = f(x)$ is shown above. It has a local maximum at $x = -1$ and a local minimum at $x = 1$ the curve asymptotes to $y = 3$. Draw neat sketches of the following.

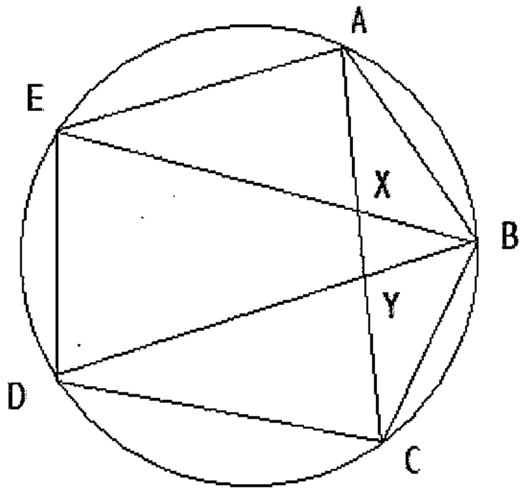
(i) $y = \ln f(x)$ 2

(ii) $y = e^{f(x)}$ 2

(iii) $y = f'(x)$ 2

(iv) $y = f(\frac{1}{x})$ 2

Question 3 (continued)



b) The pentagon ABCDE is inscribed inside the circle, with BA = BC. The diagonal AC meets the diagonals BE and BD at X and Y respectively.

(i) Show that $\angle BCA = \angle BEC$ 1

(ii) Prove that EDYX is a cyclic quadrilateral 3

c) The ellipse E has equation $\frac{x^2}{16}+\frac{y^2}{9}=1$ The point $P(x_1, y_1)$ lies on the ellipse

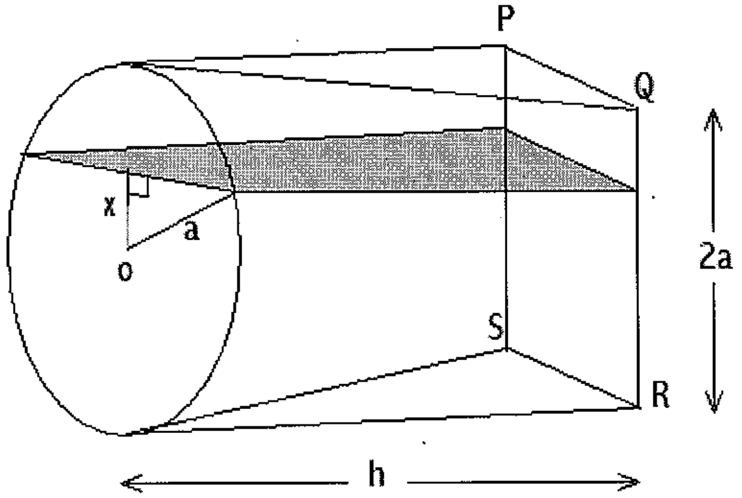
(i) Find the equation of the tangent at P to E. 2

(ii) The chord of contact to the ellipse has equation $\frac{x_0 x}{16}+\frac{y_0 y}{9}=1$

(there is no need to derive this). Show that if the chord passes through the focus, (x_0, y_0) lies on the directrix. 1

Question 4 (15 Marks)

a) The diagram below shows a solid of length h. It has a circular end of radius a units and the other end is the square PQRS of side $2a$. Horizontal cross-sections parallel to the base of the solid are taken at x as marked on the diagram.

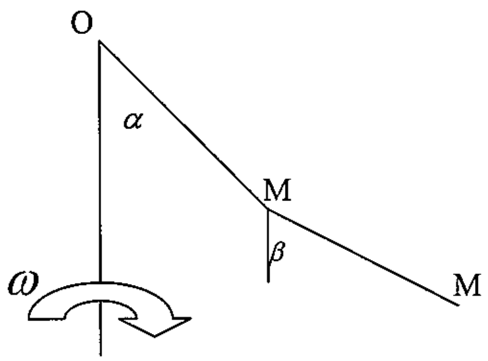


(i) Find the area of the slice at x. 2

(ii) Express the volume of the solid as an integral 1

(iii) Find the volume of the solid in (ii) 3

b) A particle hangs by a light inextensible string of length a from a fixed point O and a second particle of equal mass hangs from the first by an equal string. The whole system moves with with constant angular speed ω about the vertical through O, the upper and lower strings making constant angles α and β respectively with the vertical.



(i) Resolve forces vertically and horizontally for both masses m 4

(ii) Show that $\tan\beta = p(\sin\alpha + \sin\beta)$ 3

(iii) Show that $\tan\alpha = p(\sin\alpha + 0.5\sin\beta)$ 2

Where $p = \dfrac{a\omega^2}{g}$

Question 5 (15 Marks)

a) Given the locus $\frac{x^2}{9-k}+\frac{y^2}{4-k}=1$, with k < 4 Find:

(i) The eccentricity. 2

(ii) The coordinates of the foci. 1

(iii) The equation of the directrices. 1

b) Given the locus $\frac{x^2}{9-k}+\frac{y^2}{4-k}=1$, with 4 < k < 9 Find:

(i) The eccentricity. 2

(ii) The coordinates of the foci. 1

(iii) The equation of the directrices. 1

c) A vehicle rounds a banked track of radius 600m, inclined at an angle of θ to the horizontal. When the car travels at a speed of 10m/s the friction force up the track is equal to the friction force down the track when the vehicle travels at 20m/s. Gravity g = 9.8m/s.

(i) Resolve the forces in mutually perpendicular directions at 10m/s. 2

(ii) Resolve the forces in mutually perpendicular directions at 20m/s. 2

(iii) Find the angle θ at which the track is banked. 2

(iv) Find the speed the car travels to experience no friction force. 1

Hint the mutually perpendicular directions may be parallel and perpendicular to the track or perpendicular and horizontal.

Question 6. (15 Marks)

a) If α, β and γ are the roots of the cubic equation $x^3 - px^2 + qx - r = 0$. Find in terms of p , q and r

(i) $\alpha + \beta + \gamma$ 1

(ii) $\alpha^2 + \beta^2 + \gamma^2$ 2

(iii) $\alpha^3 + \beta^3 + \gamma^3$ 2

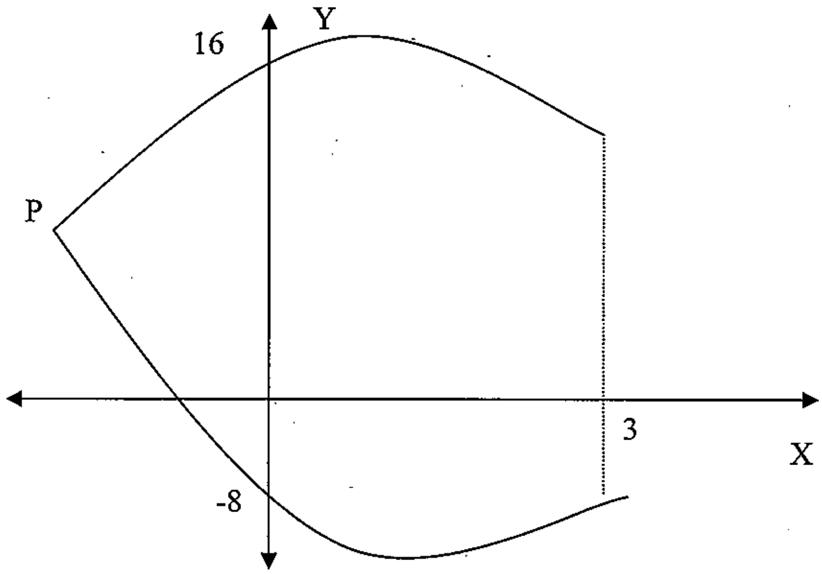
(iv) Hence find a solution to the set of equations 4

$X + Y + Z = -1$

$X^2 + Y^2 + Z^2 = 5$

$X^3 + Y^3 + Z^3 = -7$

b)



The region bounded by the curves $y = 16 - x^2$ and $y = x^2 - 2x - 8$ and the line x = 3 is rotated about the line x = 3. The point P is the point of intersection of the curves $y = 16 - x^2$ and $y = x^2 - 2x - 8$ in the second quadrant.

(i) Find the coordinates of P. 1

(ii) Use the method of cylindrical shells to express the volume of the resulting solid of revolution as an integral 3

(iii) Evaluate the integral in (ii) 2

Question 7 (15 Marks)

a) The sequence of numbers $u_1, u_2, u_3, u_4, ...u_n$ is defined as follows

$u_1 = 1,\ u_2 = 1$ and $u_n = u_{n-1} + u_{n-2}$ for $n > 3$

Prove that for every positive integer n , $u_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$

Where α and β $(\alpha > \beta)$ are the roots of $x^2 - x - 1 = 0$

(Hint in step 1. prove true for n = 1 and n = 2) 4

b) (i) Show that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ 2

(ii) Hence find the value of $\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin x + \cos x}\, dx$ 2

c) If n is a positive integer prove $\left(\frac{1+i\tan\theta}{1-i\tan\theta}\right)^n = \frac{1+i\tan n\theta}{1-i\tan n\theta}$ 2

d) The ellipse E has equation $2y^2 - 3xy + 2x^2 = 14$

(i) Using implicit differentiation or otherwise find an expression for the first derivative. 1

(ii) Find the coordinates of any turning points of E 2

(iii) Find the coordinates of any vertical tangents to E 2

Question 8 (15 Marks)

a) (i) Prove that $\dfrac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}=\sin\theta+i\cos\theta$ 2

(ii) Hence find the smallest value of θ such that

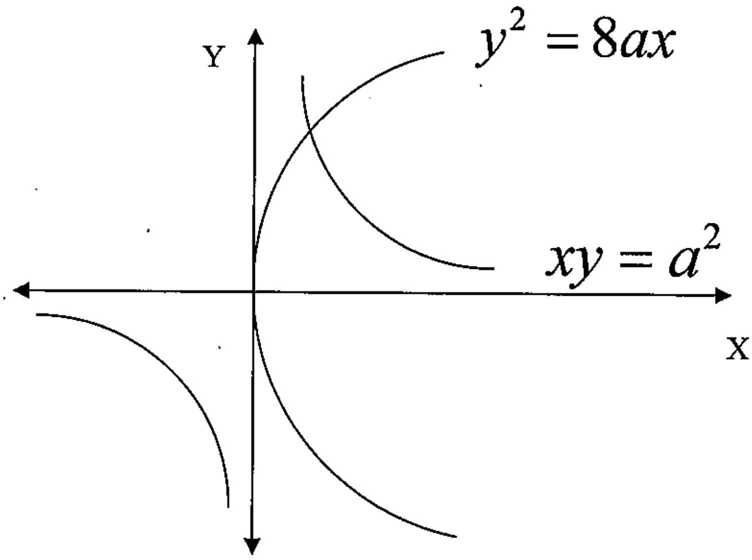
$(1+\sin\theta+i\cos\theta)^5+i(1+\sin\theta-i\cos\theta)^5=0$ 2

b) Let $I_n=\int_0^1 x(1-x)^n\,dx \quad n=0,1,2,3\ \ldots$

(i) Show that $I_n=\dfrac{n}{n+2}I_{n-1}$ 3

(ii) Show that $I_n=\dfrac{1}{2(^{n+2}C_2)}$ 2

c)



Given the hyperbola $xy=a^2$ (H)

and the parabola $y^2=8ax$ (P)

(i) Find the coordinates of A the point of intersection of H and P 1

(ii) If x + y + k = 0 is the common tangent to H and P, find k 2

(iii) Find the points of contact B on P and C on H 2

(iv) Show that AB is a tangent to H at A 1