GIRRAWEEN HIGH SCHOOL

2007

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Marks may be deducted for careless or badly arranged work

Total marks – 120

- Attempt Questions 1 – 8
- All questions are of equal value
- Start a separate piece of paper for each question.
- Put you name and the question number at the top of each sheet.

Total marks – 120
Attempt Questions 1 – 8
All questions are of equal value

Answer each question on a SEPARATE piece of paper clearly marked Question 1, Question 2, etc. Each piece of paper must show your name.

Question 1 ***(15 marks)*** Use a *separate* piece of paper — ***Marks***

a) Find $\int \frac{\sin\theta d\theta}{\cos^5\theta}$ — *2*

b) By completing the square, evaluate $\int_{-1}^{0} \frac{dx}{\sqrt{1-2x-x^2}}$ — *3*

c) (i) Find A, B and C if $\frac{16}{(x^2+4)(2-x)} = \frac{Ax+B}{(x^2+4)} + \frac{C}{(2-x)}$ — *2*

(ii) Hence find $\int \frac{16}{(x^2+4)(2-x)} dx$ — *2*

d) Use the substitution $x = 3\sin\theta$ to find $\int \frac{x^2}{\sqrt{9-x^2}} dx$ — *3*

e) Using $t = \tan\frac{x}{2}$ find $\int \frac{dx}{1+\sin x}$ — *3*

Question 2 ***(15 marks)*** Use a *separate* piece of paper

a) Let $z = 2 - i$ and $\omega = 1 + 3i$, find;

(i) $z\overline{\omega}$ — *1*

(ii) $\frac{2}{z}$ — *1*

b) Let $\alpha = \frac{4i}{-1+i\sqrt{3}}$

(i) Express α in modulus argument form — *2*

(ii) Express α^5 in modulus argument form — *2*

(iii) Hence express α^5 in the form $x + iy$ — *1*

c) Sketch the region on the Argand diagram where the inequalities — *3*

$$\frac{\pi}{4} \leq \arg(z-i) \leq \frac{3\pi}{4} \quad \text{and} \quad |z-i| \leq 2$$

hold simultaneously

Question 2 ***(continued)*** ***Marks***

d) The point A represents the real number $z_1 = 1$. The point B represents the complex number $z_2 = (1+\sqrt{3}) + i$. If $ABCD$ is a square in anti-clockwise rotational order, find;

(i) the complex number z_3 represented by the point D. *2*

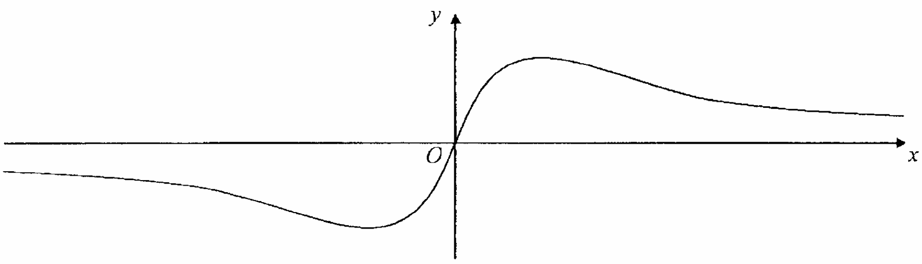
(ii) the complex number z_4 represented by the point C. *2*

(iii) the complex number z_5 represented by the point of intersection of the diagonals AC and BD. *1*

Question 3 ***(15 marks)*** Use a *separate* piece of paper

a) Sketch $y = \dfrac{3x^2}{4-x^2}$ showing all asymptotes. *3*

b) The diagram shows the graph of $y = f(x)$ which has range $-1 \le y \le 1$



Draw separate one-third page sketches of the graphs of the following;

(i) $y = |f(x)|$ *1*

(ii) $y = f(x+1)$ *1*

(iii) $y = [f(x)]^2$ *2*

(iv) $y = \dfrac{1}{f(x)}$ *2*

(v) $y = e^{f(x)}$ *2*

c) It is know that $P(x) = x^4 - 4x^3 + 5x^2 + ax + b$ is divisible by $(x-2)^2$

(i) Find the values of a and b. *3*

(ii) Factorise $P(x)$ into linear factors *1*

Question 4 ***(15 marks)*** Use a *separate* piece of paper | ***Marks***

a) A curve has the equation $x^3 + 2xy - 4y^2 = 10$. *2*

Find an expression for $\frac{dy}{dx}$ as a function of x and y.

b) A cylindrical hole of radius 1 cm is bored through the centre of a sphere of radius 3 cm. Using the method of cylindrical shells, calculate the exact volume of the sphere that remains. *4*

c) For the conic $\frac{x^2}{25} + \frac{y^2}{16 - \lambda} = 1$ find ;

(i) the values of λ that makes the conic an ellipse with the foci on the x axis. *2*

(ii) the values of λ that makes the conic a rectangular hyperbola. *1*

(iii) the coordinates of the foci and the equations of the directrices when $\lambda = 0$ *2*

d) How many different ways are there of seating four married couples at a circular table with men and women in alternate positions and no wife next to her husband? *4*

Question 5 ***(15 marks)*** Use a *separate* piece of paper

a) Suppose α, β and γ are the three roots of the polynomial equation $x^3 + x + 12 = 0$

(i) Find $\alpha^2 + \beta^2 + \gamma^2$ *2*

(ii) Hence explain why only one of the roots is real *1*

(iii) The real root is denoted by α. Prove that $-3 < \alpha < -2$ *1*

(iv) Hence prove that the modulus of each of the other roots lies between 2 and $\sqrt{6}$ *3*

b) A woman of mass M kg jumps vertically (feet first) from a rock ledge into a river below.

When she is falling at v m/s, she encounters air resistance equal to $\frac{Mv}{10}$ Newtons.

She hits the water at a speed of V m/s.

Let x be the displacement below the rock ledge at time t seconds after jumping.

(i) Show that $\ddot{x} = g - \frac{v}{10}$, where g is the acceleration due to gravity *1*

(ii) If it takes one second for her feet to hit the water, using $g = 10 \text{ m/s}^2$ show that; *3*

$$V = 100\left(1 - e^{-\frac{1}{10}}\right)$$

(iii) Find the height of the rock ledge above the water, to the nearest $0 \cdot 1$ metre *4*

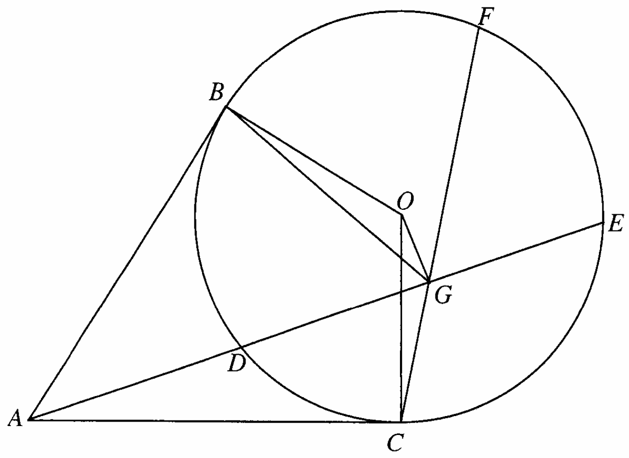
<u>Question 6</u> ***(15 marks)*** Use a *separate* piece of paper **Marks**

a) By considering the series $1+t+t^2+t^3+\cdots+t^n$, or otherwise;

(i) sum the series $1+2t+3t^2+4t^3+\cdots+nt^{n-1}$ *2*

(ii) hence evaluate $1+2\times2+3\times2^2+4\times2^3+\cdots+20\times2^{19}$ *1*

b)



In the diagram, AB and CA are the tangents from A to the circle with centre O, meeting the circle at B and C.

ADE is a secant of the circle. G is the midpoint of DE. CG produced meets the circle at F.

(Note: on page 10 there is a copy of the above diagram that you can use, please detach and include with your solutions)

(i) Show that ***ABOC*** is a cyclic quadrilateral *2*

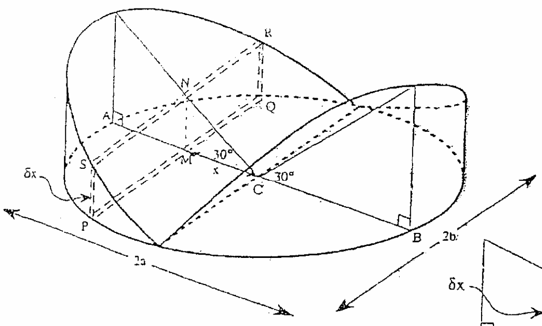
(ii) Show that ***AOGC*** is a cyclic quadrilateral *2*

(iii) Hence prove that $BF \parallel AE$ *3*

Question 6 ***(continued)*** ***Marks***

c) The roof of a sports stadium has an elliptical base with a major axis of length $2a$ and minor axis of length $2b$. The two identical sloping tops are inclined at 30° to the base.

PQRS represents a rectangular cross-section of thickness δx taken x units from the centre O of the ellipse.



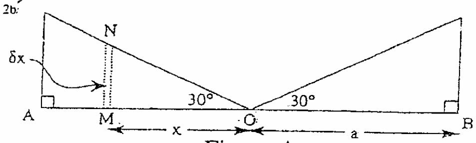


Figure A

Figure A shows a side view of the stadium if sliced in half along *AB*

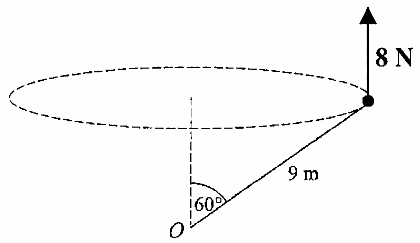
(i) Find the height *MN* of the rectangular cross-section *PQRS* *1*

(ii) Show that the area *A* of the rectangle *PQRS* is given by $A = \dfrac{2b}{a\sqrt{3}} x\sqrt{a^2 - x^2}$ 2

(iii) Calculate the volume of the stadium roof. 2

Question 7 *(15 marks)* Use a *separate* piece of paper **Marks**

a)



A toy aircraft of mass $0 \cdot 5$ kg is attached to one end of a string of length 9 m. The other end of the string is attached to a fixed point O and moves with constant angular velocity.

The string is taut, and makes an angle of 60° with the upward vertical at O.

In a simplified model of the motion, the aircraft is treated as a particle and the force of the air on the aircraft is taken to act vertically upwards with magnitude 8 Newtons. (Use $g = 10 \text{ m/s}^2$)

(i) Resolve the forces on the aircraft in the horizontal and vertical directions 2

(ii) Find the tension in the string 1

(ii) Find the speed of the aircraft in m/s. 2

b) $P\left(cp, \frac{c}{p}\right)$ is a point on the hyperbola $xy = c^2$

(i) Show that the equation of the tangent at P has the equation $x + p^2 y = 2cp$ 2

(ii) The tangent at P cuts the x axis and y axis at A and B respectively. Find the coordinates of A and B. 1

(iii) Q is the fourth vertex of the rectangle $OAQB$. Show that the locus of Q is another rectangular hyperbola. 2

c) (i) Let $I_n = \int \tan^n \theta \, d\theta$, show that $I_n = \frac{1}{n-1}\tan^{n-1}\theta - I_{n-2}$ 3

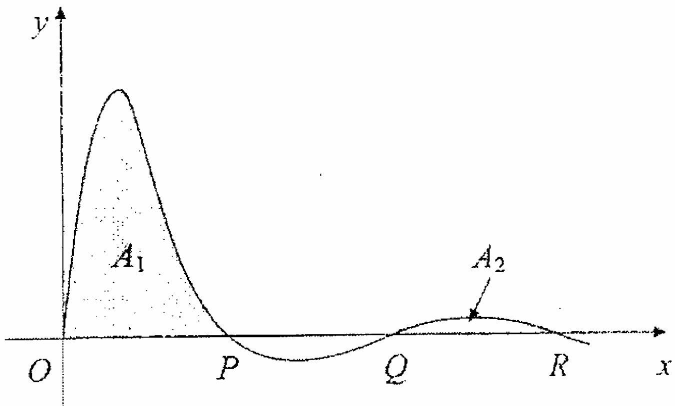
(ii) Hence find the exact value of $\int_0^{\frac{\pi}{4}} \tan^4 \theta \, d\theta$ 2

Question 8 *(15 marks)* Use a *separate* piece of paper **Marks**

a) (i) Given that $y = x\sin x$, find $\dfrac{d^2y}{dx^2}$ 1

(ii) Prove by induction that $\dfrac{d^{2n}y}{dx^{2n}} = (-1)^n(x\sin x - 2n\cos x)$ 3

b)



The diagram shows a sketch of part of the curve C with equation

$$y = e^{-x}\sin x, \quad x \geq 0$$

(i) Find the coordinates of the points P, Q and R where C cuts the x axis. 1

(ii) Use integration by parts to show that; 3

$$\int e^{-x}\sin x\,dx = -\frac{1}{2}e^{-x}(\sin x + \cos x) + c$$

(iii) The terms $A_1, A_2, \ldots, A_n$ represent areas between C and the x axis for successive portions of C where y is positive. The areas represented by A_1 and A_2 are shown in the diagram. 2

Show that $A_n = \dfrac{1}{2}\left(e^{(1-2n)\pi} + e^{(2-2n)\pi}\right)$

(iv) Show that $A_1 + A_2 + A_3 + \cdots$ is a geometric series and that $S_\infty = \dfrac{e^\pi}{2(e^\pi - 1)}$ 3

(v) Given that $\int_0^\infty e^{-x}\sin x\,dx = \dfrac{1}{2}$, find the exact value of 2

$$\int_0^\infty \left|e^{-x}\sin x\right| dx$$

Extension 2 Trial Solutions 2007

Question 1 (15)

a) $\int \frac{\sin\theta\, d\theta}{\cos^5\theta}$ $u = \cos\theta$, $du = -\sin\theta\, d\theta$

$= -\int \frac{du}{u^5}$

$= \frac{1}{4u^4} + c$

$= \frac{1}{4\cos^4\theta} + c$ (2)

b) $\int_{-1}^{0} \frac{dx}{\sqrt{1-2x-x^2}}$

$= \int_{-1}^{0} \frac{dx}{\sqrt{2-(x+1)^2}}$

$= \left[\sin^{-1}\left(\frac{x+1}{\sqrt{2}}\right)\right]_{-1}^{0}$

$= \sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1}0$

$= \frac{\pi}{4}$ (3)

c) (i) $(Ax+B)(2-x) + C(x^2+4) = 16$

$x=2$: $8C = 16$, $C = 2$

$x = 2i$: $2(2Ai+B)(2-2i) = 16$

$2A + B = 8$, $2A - B = 0$

$\therefore A = 2,\ B = 4$

$A = 2,\ B = 4,\ C = 2$

(ii) $\int \frac{16}{(x^2+4)(2-x)}dx = \int \left[\frac{2x}{x^2+4} + \frac{4}{x^2+4} + \frac{2}{2-x}\right]dx$

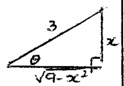
$= \log(x^2+4) + 2\tan^{-1}\frac{x}{2} - 2\log(2-x) + c$

$= \log\left[\frac{x^2+4}{(2-x)^2}\right] + 2\tan^{-1}\frac{x}{2} + c$

d) $\int \frac{x^2}{\sqrt{9-x^2}}dx$ $x = 3\sin\theta$, $dx = 3\cos\theta\, d\theta$

$= \int \frac{9\sin^2\theta \cdot 3\cos\theta\, d\theta}{3\cos\theta}$

$= \int 9\sin^2\theta\, d\theta$



$= \frac{9}{2}\int (1-\cos 2\theta)\, d\theta$

$= \frac{9}{2}\theta - \frac{9}{4}\sin 2\theta + c$

$= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + c$

$= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{1}{2}x\sqrt{9-x^2} + c$ (3)

e) $\int \frac{dx}{1+\sin x}$ $t = \tan\frac{x}{2}$, $dx = \frac{2dt}{1+t^2}$

$= \int \frac{\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}}$

$= \int \frac{2\, dt}{1+t^2+2t}$

$= \int \frac{2}{(t+1)^2}dt$

$= -\frac{2}{t+1} + c$

$= -\frac{2}{\tan\frac{x}{2}+1} + c$

Question 2 (15)

a) (i) $z\overline{w} = (2-i)(1-3i)$
$= 2-6i-i-3$
$= -1-7i$ (1)

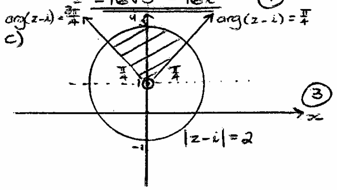
(ii) $\frac{2}{z} = \frac{2\overline{z}}{|z|^2}$
$= \frac{2(2+i)}{5}$ (1)

b) $\alpha = \frac{4i}{-1+i\sqrt{3}}$
$= \frac{4\text{cis}\frac{\pi}{2}}{2\text{cis}\frac{2\pi}{3}}$
$= 2\text{cis}\frac{-\pi}{6}$ (2)

(ii) $\alpha^5 = 2^5\text{cis}\frac{-5\pi}{6}$
$= 32\text{cis}\frac{-5\pi}{6}$ (2)

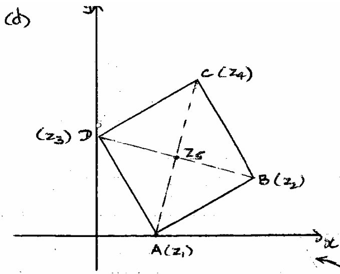
(iii) $\alpha^5 = 32\left(\cos\frac{-5\pi}{6} + i\sin\frac{-5\pi}{6}\right)$
$= 32\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$
$= -16\sqrt{3} - 16i$ (1)

c)



(3)

(d)



(i) $z_3 - z_1 = i(z_2 - z_1)$
$z_3 - 1 = i(\sqrt{3}+i)$
$= \sqrt{3}i - 1$
$z_3 = \sqrt{3}i$ (2)

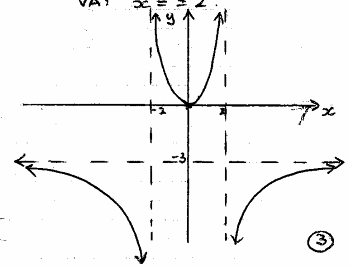
(ii) $z_4 = z_2 + z_3 - z_1$
$= (1+\sqrt{3}) + i + \sqrt{3}i - 1$
$= \sqrt{3} + (1+\sqrt{3})i$ (2)

(iii) z_5 = midpoint of AC
$= \frac{\sqrt{3}+1}{2} + \frac{\sqrt{3}+1}{2}i$ (1)

Question 3 (15)

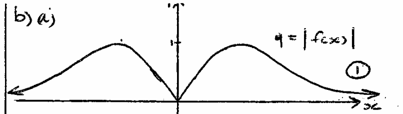
a) $y = \frac{3x^2}{4-x^2}$
$= -3 + \frac{12}{4-x^2}$

$\therefore$ HA: $y = -3$
VA: $x = \pm 2$



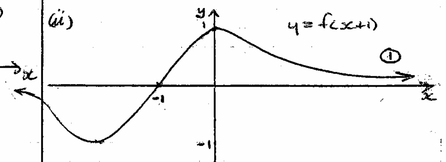
(3)

b) (i)



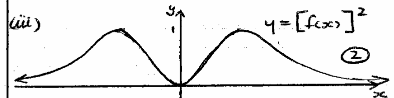
(1)

(ii)



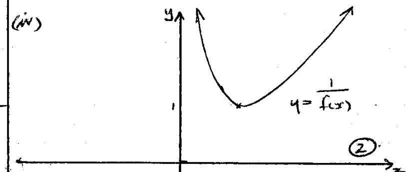
(1)

(iii)



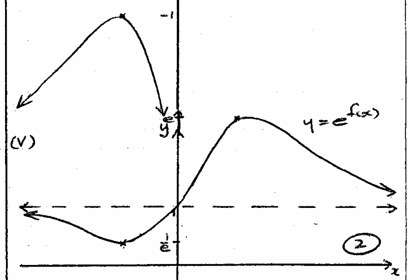
(2)

(iv)



(2)

(v)



(2)

c)(i) $P(x) = x^4 - 4x^3 + 5x^2 + ax + b$
$P'(x) = 4x^3 - 12x^2 + 10x + a$

$P(2) = 0$
$16 - 32 + 20 + 2a + b = 0$
$2a + b = -4$

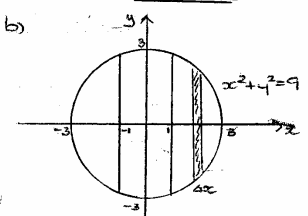
$P'(2) = 0$
$32 - 48 + 20 + a = 0$
$a = -4, \therefore b = 4$ (3)

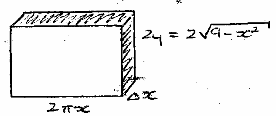
(ii) $P(x) = x^4 - 4x^3 + 5x^2 - 4x + 4$
$= (x^2 - 4x + 4)(x^2 + 1)$
$= (x-2)^2(x+i)(x-i)$ (1)

Question 4 (15)

a) $x^3 + 2xy - 4y^2 = 10$
$3x^2 + (2x)\frac{dy}{dx} + (y)(2) - 8y\frac{dy}{dx} = 0.$
$(2x - 8y)\frac{dy}{dx} = -3x^2 - 2y$
$\frac{dy}{dx} = \frac{3x^2 + 2y}{8y - 2x}$ (2)

b)





$A(x) = 4\pi x\sqrt{9 - x^2}$
$\Delta V = 4\pi x\sqrt{9 - x^2}\,\Delta x$
$V = \lim_{\Delta x \to 0} \sum_{x=1}^{3} 4\pi x\sqrt{9 - x^2}\,\Delta x$
$= 2\pi \int_1^3 2x\sqrt{9 - x^2}\,dx$
$= -2\pi\left[\frac{2}{3}(9 - x^2)^{\frac{3}{2}}\right]_1^3$
$= -\frac{4\pi}{3}(0 - 8\sqrt{8})$
$= \frac{64\sqrt{2}\pi}{}$ u^3 (4)

c) $0 < 16 - \lambda < 25$
$-16 < -\lambda < 9$
$-9 < \lambda < 16$ (2)

(ii) $16 - \lambda = -25$
$\lambda = 41$ (1)

(iii) $\frac{x^2}{25} + \frac{y^2}{16} = 1$
$a^2 = 25$ $\quad b^2 = a^2(1 - e^2)$
$a = 5$ $\quad 16 = 25(1 - e^2)$
$1 - e^2 = \frac{16}{25}$
$e^2 = \frac{9}{25}$
$e = \frac{3}{5}$ (2)
$\therefore$ foci $(\pm 3, 0)$ directrices $x = \pm\frac{25}{3}$

d) Place men first = 3!
A, B, C, D (Ma)
Wife A has 2 spots
this leaves 1 spot for each of the other wives

Ways = $3! \times 2 \times 1 \times 1 \times 1$
$= 12$ (4)

Question 5 (15)

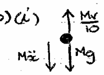
a)(i) $x^3 + x + 12 = 0$
$\sum\alpha^2 = (\sum\alpha)^2 - 2\sum\alpha\beta$
$= 0^2 - 2(1)$
$= -2$ (2)

(ii) as $\sum\alpha^2 < 0$ there must be an imaginary root.
However when coefficients are real, roots appear in conjugate pairs
ie there are two imaginary roots
$\therefore$ there is one real root (1)

(iii) $P(-3) = -27 - 3 + 12$ $\quad P(-2) = -8 - 2 + 12$
$= -18 < 0$ $\quad = 2 > 0$
as polynomial is continuous
$-3 < \alpha < -2$ (1)

(iv) $\alpha\beta\gamma = -12$
$4 < \beta\gamma < 6$
$\beta\gamma$ are conjugates
$z\bar{z} = |z|^2$
$\therefore 4 < |z|^2 < 6$
$2 < |z| < \sqrt{6}$ (3)

b)(i)



$M\ddot{x} = Mg - \frac{Mv}{10}$
$\ddot{x} = g - \frac{v}{10}$ (1)

(ii) $\frac{dv}{dt} = \frac{10g - v}{10}$
$t = -10\int_0^V \frac{-dv}{10g - v}$

when $t = 1$, $v = V$
$1 = -10\int_0^V \frac{-dt}{10g - v}$
$-\frac{1}{10} = [\log(10g - v)]_0^V$
$= \log\left(\frac{10g - V}{10g}\right)$
$e^{-\frac{1}{10}} = \frac{10g - V}{10g}$
$10g\,e^{-\frac{1}{10}} = 10g - V$
$V = 10g\left(1 - e^{-\frac{1}{10}}\right)$ (3)
$V = 100\left(1 - e^{-\frac{1}{10}}\right)$ $(\because g = 10)$

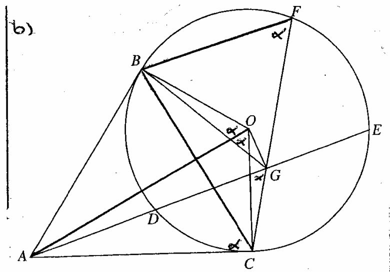
(iii) $v\frac{dv}{dx} = \frac{10g - v}{10}$
$x = 10\int_0^V \frac{v\,dv}{10g - v}$
$x = -10\int_0^V \left(1 - \frac{10g}{10g - v}\right)dv$
$= -10\left[v + 10g\log(10g - v)\right]_0^V$
$= -10\left(V + 10g\log\left(\frac{10g - V}{10g}\right)\right)$
$= -10\left(100(1 - e^{-\frac{1}{10}}) + 100(-\frac{1}{10})\right)$
$= 4.8$m (4)

Question 6 (15)

a)(i) $1 + t + t^2 + t^3 + \ldots + t^n = \frac{t^n - 1}{t - 1}$
$1 + 2t + 3t^2 + \ldots + nt^{n-1} = \frac{(t-1)(nt^{n-1}) - (t^n)(1)}{(t-1)^2}$
$= \frac{(n-1)t^n - nt^{n-1} + 1}{(t-1)^2}$ (2)

(ii) $t = 2$, $n = 20$
$1 + 2\times 2 + 3\times 2^2 + \ldots + 20\times 2^{19}$
$= 19\times 2^{20} - 20\times 2^{19} + 1$
$= 9\,437\,185$ (1)

b)



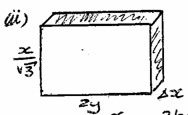
(i) $\angle OBA = 90°$ (radius $\perp$ tangent)
$\angle OCA = 90°$ (")
$\therefore$ ABOC is cyclic quadrilateral (opposite $\angle$'s supplementary) (2)

(ii) $\angle OGA = 90°$ ($\perp$ centre bisects chord)
$\therefore \angle OGA = \angle OCA$
AOGC is cyclic quadrilateral ($\angle$'s in same segment =) (2)

(iii) Let $\angle BFC = \alpha$
$\angle BOC = 2\alpha$ ($\angle$ at centre twice $\angle$ at circumference on same arc)
$\angle BCA = \alpha$ ($\angle$ in alternate segment)
$\therefore \angle BOA = \alpha$ ($\angle$'s in same segment = in ABOC)
$\angle BOC = \angle AOB + \angle AOC$ (common $\angle$)
$2\alpha = \alpha + \angle AOC$
$\angle AOC = \alpha$
$\angle AOC = \angle AGC$ ($\angle$'s in same segment = in AOGC)
$\therefore \angle AGC = \alpha$
$\angle AGC = \angle BFC = \alpha$ (3)
$\therefore$ BF $\parallel$ AE (corresponding $\angle$'s =)

c) $\frac{MN}{x} = \tan 30°$
$MN = \frac{x}{\sqrt{3}}$ (1)

(ii)



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
$a^2y^2 = a^2b^2 - b^2x^2$
$y^2 = \frac{b^2(a^2 - x^2)}{a^2}$

$A(x) = \frac{x}{\sqrt{3}} \times \frac{2b\sqrt{a^2 - x^2}}{a}$
$= \frac{2b}{a\sqrt{3}}\,x\sqrt{a^2 - x^2}$ (2)

(iii) $\Delta V = \frac{2b}{a\sqrt{3}} x\sqrt{a^2-x^2}\,\Delta x$

$V = \lim_{\Delta x \to 0} \sum_{x=-a}^{a} \frac{2b}{a\sqrt{3}} x\sqrt{a^2-x^2}\,\Delta x$

$= \frac{2b}{a\sqrt{3}} \int_0^a 2x\sqrt{a^2-x^2}\,dx$

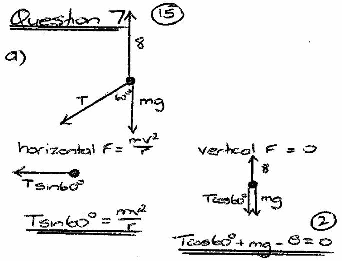
$= \frac{-2b}{a\sqrt{3}} \left[\frac{2}{3}(a^2-x^2)^{\frac{3}{2}}\right]_0^a$

$= \frac{4b}{3a\sqrt{3}}(a^3)$

$= \frac{4a^2b}{3\sqrt{3}}$ units^3 (2)

Question 7 (15)

a)



$T\sin 60° = \frac{mv^2}{r}$

$T\cos 60° + mg - 8 = 0$ (2)

(ii) $\frac{T}{2} = 8 - (0.5)(10)$

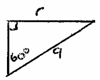
$T = 6\ N$ (1)

(iii) $\frac{\sqrt{3}T}{2} = \frac{mv^2}{r}$

$\frac{\sqrt{3}(6)}{2} = \frac{(0.5)v^2}{\frac{9\sqrt{3}}{2}}$

$v^2 = 3\sqrt{3} \times \frac{9\sqrt{3}}{2} \times 2$

$= 81$

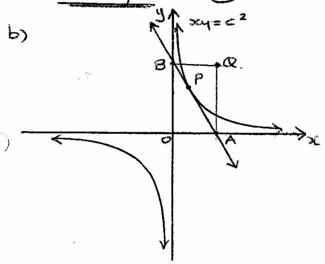


$\frac{r}{9} = \sin 60°$

$r = \frac{9\sqrt{3}}{2}$

$v = 9\ m/s$ (2)

b)



$y = \frac{c^2}{x}$

$\frac{dy}{dx} = \frac{-c^2}{x^2}$

when $x = cp$, $\frac{dy}{dx} = -\frac{1}{p^2}$

$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$

$p^2y - pc = -x + cp$

$x + p^2y = 2cp$ (2)

(ii) A: $y = 0$ ∴ $x = 2cp$ B: $x = 0$

$p^2y = 2cp$

$y = \frac{2c}{p}$

$A(2cp, 0)$ B: $(0, \frac{2c}{p})$ (1)

(iii) $Q\left(2cp, \frac{2c}{p}\right)$

$xy = 2cp \times \frac{2c}{p}$

$xy = 4c^2$

∴ locus of Q is $xy = 4c^2$ (2)

c)(i) $I_n = \int \tan^n\theta\,d\theta$

$= \int \tan^{n-2}\theta(\sec^2\theta - 1)\,d\theta$

$= \int \tan^{n-2}\theta\sec^2\theta\,d\theta - \int \tan^{n-2}\theta\,d\theta$ $u = \tan\theta$, $du = \sec^2\theta\,d\theta$

$= \int u^{n-2}\,du - I_{n-2}$

$= \frac{u^{n-1}}{n-1} - I_{n-2}$

$= \frac{1}{n-1}\tan^{n-1}\theta - I_{n-2}$ (3)

(ii) $\int_0^{\frac{\pi}{4}} \tan^4\theta = I_4$

$= \left[\frac{1}{3}\tan^3\theta\right]_0^{\frac{\pi}{4}} - I_2$

$= \left[\frac{1}{3}\tan^3\theta - \tan\theta\right]_0^{\frac{\pi}{4}} + I_0$

$= \left[\frac{1}{3}\tan^3\theta - \tan\theta\right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} d\theta$

$= \left[\frac{1}{3}\tan^3\theta - \tan\theta + \theta\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{3} - 1 + \frac{\pi}{4}$

$= \frac{\pi}{4} - \frac{2}{3}$

$= \frac{3\pi - 8}{12}$ (2)

Question 8 (15)

a)(i) $y = x\sin x$

$\frac{dy}{dx} = (x)(\cos x) + (\sin x)(1)$

$= x\cos x + \sin x$

$\frac{d^2y}{dx^2} = (x)(-\sin x) + (\cos x)(1) + \cos x$

$= 2\cos x - x\sin x$ (1)

(ii) Prove true for $n = 1$

LHS $= \frac{d^2y}{dx^2}$ RHS $= (-1)(x\sin x - 2\cos x)$

$= 2\cos x - x\sin x$ $= 2\cos x - x\sin x$

∴ LHS = RHS

Hence the result is true for $n = 1$

Assume the result is true for $n = k$

ie $\frac{d^{2k}y}{dx^{2k}} = (-1)^k(x\sin x - 2k\cos x)$

Prove the result is true for $n = k+1$

ie $\frac{d^{2k+2}y}{dx^{2k+2}} = (-1)^{k+1}(x\sin x - 2(k+1)\cos x)$

Proof:

$\frac{d^{2k+1}y}{dx^{2k+1}} = (-1)^k\left[(x)(\cos x) + \sin x + 2k \ldots\right]$

$= (-1)^k(x\cos x + (2k+1)\sin x)$

$\frac{d^{2k+2}y}{dx^{2k+2}} = (-1)^k\left[(x)(-\sin x) + (\cos x)(1) + (2k+1)\cos x\right]$

$= (-1)^k(-x\sin x + (2k+2)\cos x)$

$= (-1)^{k+1}(x\sin x - 2(k+1)\cos x)$

Hence the result is true for $n = k+1$ if it is also true for $n = k$

Hence the result is true for all positive integral values of n by induction. (3)

b)(i) $e^{-x}\sin x = 0$

$\sin x = 0$

$x = 0, \pi, 2\pi, \ldots$

∴ $P(\pi, 0)$, $Q(2\pi, 0)$, $R(3\pi, 0)$ (1)

(ii) $\int e^{-x}\sin x\,dx$

$u = e^{-x}$ $v = -\cos x$

$du = -e^{-x}dx$ $dv = \sin x\,dx$

$= -e^{-x}\cos x - \int e^{-x}\cos x\,dx$

$u = e^{-x}$ $v = \sin x$

$du = -e^{-x}dx$ $dv = \cos x\,dx$

$= -e^{-x}\cos x - e^{-x}\sin x - \int e^{-x}\sin x\,dx$

∴ $2\int e^{-x}\sin x\,dx = -e^{-x}(\sin x + \cos x) + c$

$\int e^{-x}\sin x\,dx = -\frac{1}{2}e^{-x}(\sin x + \cos x) + C$ (3)

(iii) $A_1 = \int_0^{\pi} e^{-x}\sin x\,dx$

$A_2 = \int_{2\pi}^{3\pi} e^{-x}\sin x\,dx$

∴ $A_n = \int_{(2n-2)\pi}^{(2n-1)\pi} e^{-x}\sin x\,dx$

$= -\frac{1}{2}\left[e^{-x}(\sin x + \cos x)\right]_{(2n-2)\pi}^{(2n-1)\pi}$

$= -\frac{1}{2}\left(-e^{-(2n-1)\pi} - e^{-(2n-2)\pi}\right)$

$= \frac{1}{2}\left(e^{(2-2n)\pi} + e^{(1-2n)\pi}\right)$ (2)

(iv) $A_1 = \frac{1}{2}(e^{-\pi} + 1)$

$A_2 = \frac{1}{2}(e^{-3\pi} + e^{-2\pi})$

$A_3 = \frac{1}{2}(e^{-5\pi} + e^{-4\pi})$

∴ geometric series $r = e^{-2\pi}$

$S_\infty = \frac{\frac{1}{2}(e^{-\pi} + 1)}{1 - e^{-2\pi}}$

$= \frac{e^{\pi} + e^{2\pi}}{2(e^{2\pi} - 1)}$

$= \frac{e^{\pi}(1 + e^{\pi})}{2(e^{\pi}+1)(e^{\pi} - 1)}$

$= \frac{e^{\pi}}{2(e^{\pi} - 1)}$ (3)

(v) A_n = areas above x axis

Let $A_1 + A_2 + A_3 + \ldots = X$

a_n = areas under x axis

$a_1 + a_2 + a_3 + \ldots = Y$

∴ $\int_0^{\infty} e^{-x}\sin x\,dx = X - Y$

$\frac{1}{2} = \frac{e^{\pi}}{2(e^{\pi} - 1)} - Y$

$Y = \frac{e^{\pi} - (e^{\pi} - 1)}{2(e^{\pi} - 1)}$

$= \frac{1}{2(e^{\pi} - 1)}$

$\int_0^{\infty} |e^{-x}\sin x|\,dx = X + Y$

$= \frac{e^{\pi}}{2(e^{\pi} - 1)} + \frac{1}{2(e^{\pi} - 1)}$

$= \frac{e^{\pi} + 1}{2(e^{\pi} - 1)}$ (2)