GIRRAWEEN HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2002

MATHEMATICS

EXTENSION 2

Time allowed - Three hours
(Plus 5 minutes' reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are supplied
- Board-approved calculators may be used.
- Each question attempted is to be returned on a *separate* piece of paper clearly marked Question 1 , Question 2 , etc. Each piece of paper must show your name.
- You may ask for extra pieces of paper if you need them.

QUESTION 1

a) Find $\int \frac{dx}{\sqrt{x^2+2x+5}}$ 2

b) (i) Find real numbers a, b and c such that

$$\frac{3x}{(x+1)(x^2+2x+4)} \equiv \frac{a}{x+1} + \frac{bx+c}{x^2+2x+4}$$ 2

(ii) Find $\int \frac{3x}{(x+1)(x^2+2x+4)}dx$ 2

c) Use integration by parts to find

$$\int_0^1 \tan^{-1} x \, dx$$ 3

d) Find $\int_0^{\frac{2\pi}{3}} \frac{1}{5+4\cos x}dx$ 3

e) Find $\int \sin^3 x \cos^2 x \, dx$ 3

QUESTION 2

(a) Given the two complex numbers $z = 3-4i$ and $w = 4+3i$,
find zw and $\frac{1}{w}$ in the form $x+iy$. 2

b) On separate argand diagrams draw a neat sketch of the locus specified by

(i) $z^2 - \bar{z}^2 = 4i$ 2

(ii) $\arg(z-2) = \arg z + \frac{\pi}{2}$ 2

c) If $z = \sqrt{3} + i$

(i) Find the exact value of mod z and arg z . 2

(ii) By using De Moivres theorem write $\frac{1}{z^5}$ in form x+iy . 2

d) Let P, Q, R represent the complex numbers z_1, z_2, z_3 respectively.
What geometric properties characterize triangle PQR if $z_2 - z_1 = i(z_3 - z_1)$?
Give reasons for your answer. 3

e) The polynomial $z^3 - 3z^2 + 7z - 5$ has one root equal to 1-2i .
Factorize this polynomial 2

QUESTION 3

(a) An ellipse has the equation $\frac{x^2}{25}+\frac{y^2}{16}=1$

(i) Find the eccentricity. 1

(ii) Find coordinates of the foci S, S' and equation of directrices. 2

(iii) Sketch the ellipse showing all the above features and where it crosses the coordinate axes. 1

(iv) If P is a point on the ellipse show that PS + PS' is independent of the position of P. 2

(b) Consider the hyperbola with equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

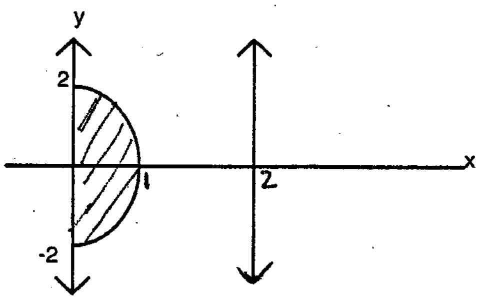
(i) Show that the equation of the tangent at the point P $(a\sec\theta, b\tan\theta)$ has the equation $bx\sec\theta - ay\tan\theta = ab$. 2

(ii) Deduce the equation of the normal at P. 2

(iii) Find A and B where the tangent and normal respectively cut the y-axis. 2

(iv) Show that AB is the diameter of the circle that passes through the foci of the hyperbola. 3

QUESTION 4



(a) A solid S is formed by rotating the region bounded by the parabola $y^2 = 4(1-x)$ and the y axis 360° about the line x = 2.

(i) By slicing perpendicular to the axis of rotation, find the exact volume of S. **4**

(ii) (*a*) Use the method of cylindrical shells to show that the volume of S is also given by $\int_0^1 8\pi(2-x)\sqrt{1-x}\,dx$. **2**

(*b*) Confirm your answer to part (i) by calculating this definite integral using the substitution u = 1-x . **3**

(b) A dome has a circular base of radius 10 metres. Each cross section of the dome perpendicular to the x-axis is a parabola, whose height is the same as the base width .

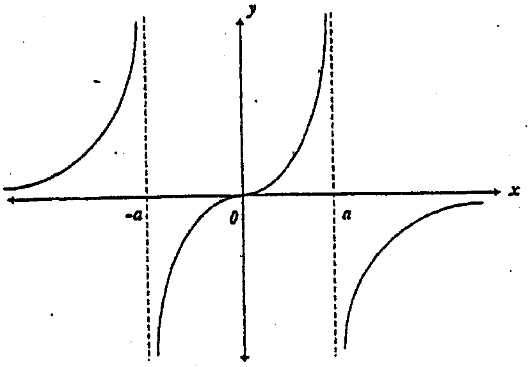
(i) Why would Simpson's rule give the exact area of the parabolic cross section? **1**

(ii) Show that the area of the parabolic cross-section is $\frac{8y^2}{3}$ square metres. **2**

(iii) Find the volume of the dome. **3**

QUESTION 5

(a) The graph of y=f(x) is shown below



Draw sketches of the following

(i) y=f(x-a) 1

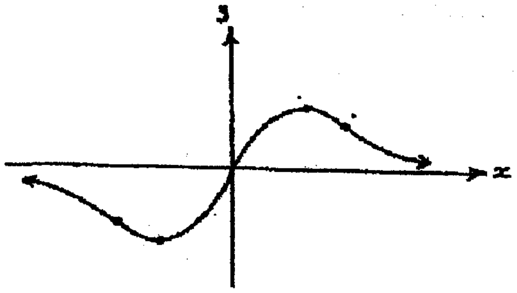
(ii) y=f′(x) 2

(iii) $y=\frac{1}{f(x)}$ 2

(iv) $y=f(x)^2$ 2

(b) Find integers a and b such that $(x+1)^2$ is a factor of x^3+4x^2+ax+b 3

(c)



The curve $y=\frac{2x}{1+x^2}$ is sketched in the diagram above

(i) Show that the equation $kx^3+(k-2)x=0$ can be written in the form $\frac{2x}{1+x^2}=kx$ 2

(ii) Using a graphical approach based on the curve $y=\frac{2x}{1+x^2}$, or otherwise, find the real values of k for which the equation $kx^3+(k-2)x=0$ has exactly 1 solution. 3

QUESTION 6

(a) A particle of mass m is projected vertically upwards under gravity

The air resistance to the motion is $-\frac{1}{100}mgv^2$ where v is the speed of the particle .

(i) Show that during the upward motion of the particle, if x is the upward vertical displacement of the particle from its projection point at time t then

$$\ddot{x} = -\frac{1}{100}g(100+v^2).$$ 2

(ii) If the speed of projection is u show that the greatest height (above the point of projection) reached by the particle is

$$\frac{50}{g}\ln\left(\frac{100+u^2}{100}\right).$$ 4

(b) Let ω be a non-real cube root of unity .

(i) Show that $1+\omega+\omega^2=0$. 1

(ii) Hence simplify $(1+\omega)^2$. 1

(iii) Show that $(1+\omega)^3 = -1$.. 1

(iv) Use part iii) to simplify $(1+\omega)^{3n}$ and hence show that

$$^{3n}C_0 - \frac{1}{2}(^{3n}C_1 + ^{3n}C_2) + ^{3n}C_3 - \frac{1}{2}(^{3n}C_4 + ^{3n}C_5) + ^{3n}C_6 - \dots\dots\dots ^{3n}C_{3n} = (-1)^n.$$ 3

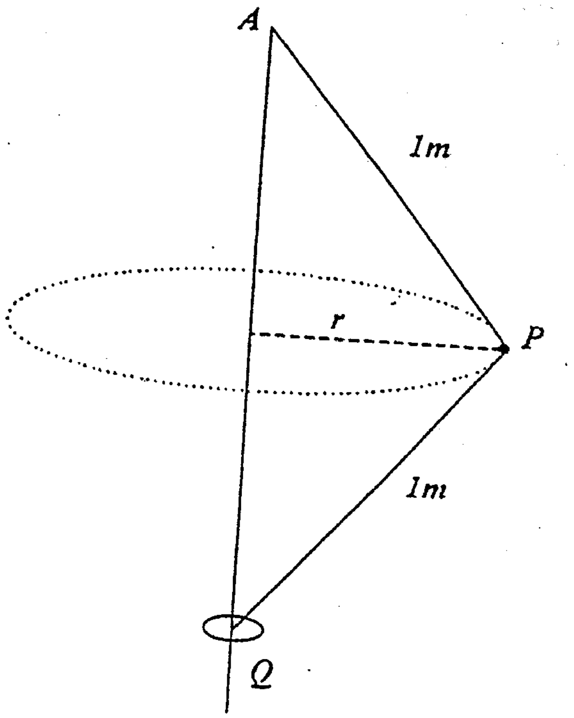
(HINT: You may assume $\mathrm{Re}(\omega) = -\frac{1}{2}$ and that $\mathrm{Re}(\omega^2) = -\frac{1}{2}$)

(c) (i) Show that for a>0 and $n \neq 0, \log_{a^n} x = \frac{1}{n}\log_a x$ 1

(ii) Hence evaluate $\log_2 3 + \log_4 3 + \log_{16} 3 + \log_{256} 3 + \dots\dots\dots$ 2

QUESTION 7

(a) A particle P, of mass 2kg, is attached by a light inelastic string of length 1m to a fixed point A as shown in the diagram below. Another string of equal length attaches P to a smooth ring Q, of mass 3kg which is free to slide on a vertical wire that passes through A. The particle P is rotating in a horizontal circle of radius r, about the vertical wire with a constant angular velocity of 2π radians per second



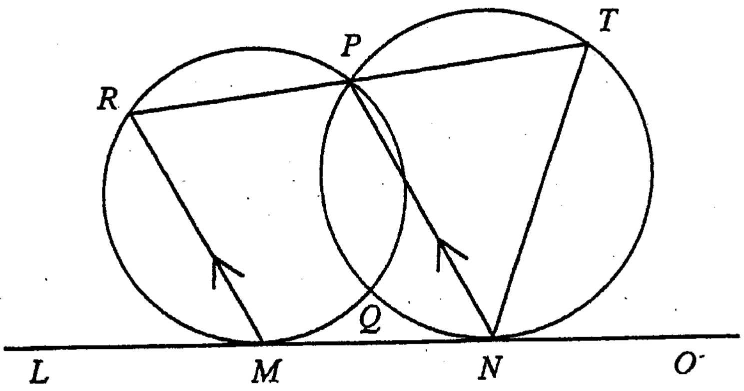
Let T_1 represent the tension in the string PQ, T_2 the tension in the string AP and θ the angle of inclination of AP to the vertical wire.

(i) Copy the above diagram onto your paper and clearly indicate on your sketch all the forces acting on P and Q. **1**

(ii) Write down the equations expressing the vertical and horizontal equilibrium of forces at points P and Q. **3**

(iii) By using the equations in (ii) evaluate $\tan\theta$ in terms of r.
Hence calculate the vertical distance h of P below A ($g=9.8ms^{-2}$) **4**

QUESTION 7 (cont)



(b) In the diagram the two circles intersect at P and Q. LMNO is a common tangent to the two circles. R is a point on one circle such that MR || NP. RP produced meets the other circle at T.

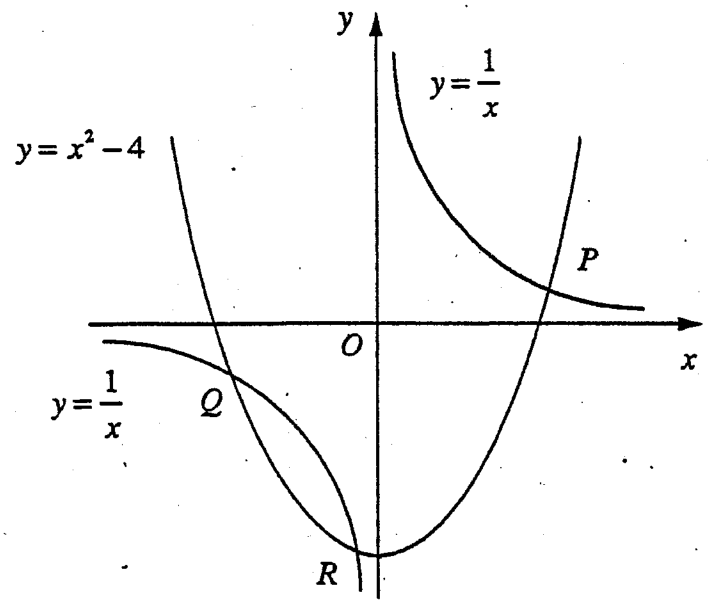
(i) Copy the diagram.

(ii) Show that MNTR is a cyclic quadrilateral. **4**

(iii) G is the point of intersection of MT and NR. The circle through the points T,R,and G is drawn. Show that the tangent to this circle at G is parallel to MN. **3**

QUESTION 8

(a)



The curves $y = x^2 - 4$ and $y = \frac{1}{x}$ intersect at the points P,Q,R where $x=\alpha, x = \beta, x = \gamma$

(i) Show that α, β, γ are the roots of the equation $x^3 - 4x - 1 = 0$ **1**

(ii) Find a polynomial equation with integer coefficients which has roots $\alpha^2, \beta^2, \gamma^2$. **2**

(iii) Find a polynomial equation with integer coefficients which has roots $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$. **2**

(iv) Hence find the numerical value of $OP^2 + OQ^2 + OR^2$. **2**

(b) Newton's Method can be used to determine numerical approximations to the real roots of the equation $x^3 = 4$.
Let $x_1 = 2, x_2, x_3, \ldots\ldots\ldots x_n, \ldots\ldots$ be a series of estimates obtained by iterative applications of Newton's method.

(i) Show that $x_{n+1} = \frac{2}{3}(x_n + \frac{2}{x_n^{\,2}})$. **2**

(ii) Show algebraically that $x_{n+1} - \sqrt[3]{4} = \frac{(x_n - \sqrt[3]{4})^2(2x_n + \sqrt[3]{4})}{3x_n^2}$. **3**

(iii) Given that $x_n > \sqrt[3]{4}$ show that $x_{n+1} - \sqrt[3]{4} < (x_n - \sqrt[3]{4})^2$. **2**

(iv) Show that x_6 is accurate to 12 decimal places. **1**