FRENSHAM

YEAR 12 TRIAL HSC EXAMINATION

2011

MATHEMATICS

EXTENSION 2

Time Allowed 3 hours +5 minutes reading time

INSTRUCTIONS:

- All questions may be attempted
- All questions are of equal value
- Show all necessary working. Marks may be deducted for careless or badly arranged work
- Start each question on a new page
- Board of Studies approved calculators may be used

Student name / number ....

Marks

Question 1 **Begin a new booklet**

(a) Find $\int \frac{x^2+1}{\sqrt{x}}\, dx$. **2**

(b) Find $\int \frac{\cos^3 x}{\sin^2 x}\, dx$ using the substitution $u = \sin x$. **3**

(c) Evaluate $\int_0^{\frac{1}{2}\log_e 3} \frac{1}{e^x + e^{-x}}\, dx$ using the substitution $u = e^x$. **3**

(d) Evaluate in simplest exact form $\int_1^e x^3 \log_e x\ dx$. **3**

(e)(i) Using the substitution $t = \tan\frac{x}{2}$, show that **2**

$$\int_0^{\frac{\pi}{2}} \frac{1}{5+5\sin x - 3\cos x}\, dx = \int_0^1 \frac{1}{4t^2+5t+1}\, dt .$$

(ii) Hence evaluate in simplest exact form $\int_0^{\frac{\pi}{2}} \frac{1}{5+5\sin x-3\cos x}\, dx$. **2**

Student name / number ....

Question 2 **Begin a new booklet** **Marks**

(a) If $z_1 = 2i$ and $z_2 = 1 + 3i$, express in the form $a + ib$ (where a and b are real)

(i) $z_1 + \overline{z}_2$. **1**

(ii) $z_1 z_2$. **1**

(iii) $\dfrac{1}{z_2}$. **1**

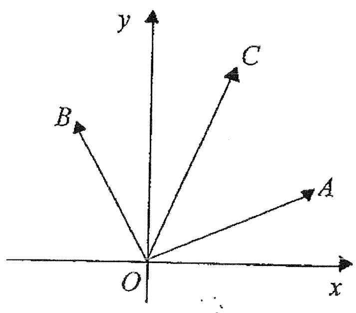
(b)(i) Express $z = 1 + i\sqrt{3}$ in modulus/argument form. **2**

(ii) Hence show that $z^{10} + 512z = 0$. **2**

(c)(i) On an Argand diagram sketch the locus of the point P representing z such that $\left| z - (\sqrt{3} + i) \right| = 1$. **2**

(ii) Find the set of possible values of $|z|$ and the set of possible principal values of $\arg z$. **2**

(d)



In the Argand diagram above, vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ represent the complex numbers z_1, z_2 and $z_1 + z_2$ respectively, where $z_1 = \cos\theta + i\sin\theta$ and $z_1 + z_2 = (1 + i)\, z_1$.

(i) Express z_2 in terms of z_1 and show that $OACB$ is a square. **2**

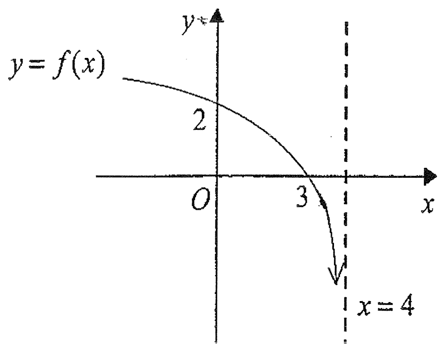
(ii) Show that $(z_1 + z_2)\overline{(z_1 - z_2)} = 2i$. **2**

Student name / number ....

Marks

Question 3 **Begin a new booklet**

(a) The diagram shows the graph of the curve $y = f(x)$. On separate diagrams, sketch the graphs of the curves listed below, showing clearly intercepts on the coordinate axes and the equations of any asymptotes:



(i) $y = \left| f(x) \right|$. **1**

(ii) $y = f\left(|x|\right)$. **1**

(iii) $y = f\left(x^2\right)$. **2**

(iv) $y = \dfrac{1}{f(x)}$. **2**

(b) $P(x)$ is an even polynomial. Show that when $P(x)$ is divided by $\left(x^2 - a^2\right)$, where $a \neq 0$, the remainder is independent of x. **3**

(c) The zeroes of $x^3 + px^2 + qx + r$ are α, β and γ (where p, q and r are real numbers).

(i) Find $\alpha\beta + \alpha\gamma + \beta\gamma$. **1**

(ii) Find $\alpha^2 + \beta^2 + \gamma^2$. **1**

(iii) Find a cubic polynomial with integer coefficients whose zeroes are $2\alpha, 2\beta$ and 2γ. **2**

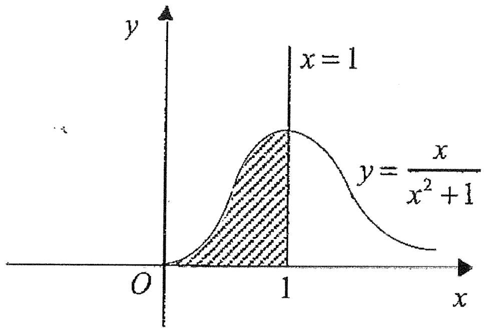
(d) If $p > 0$, and $q > 0$, and $p + q = 1$, show that $\frac{1}{p} + \frac{1}{q} \geq 4$. **2**

Student name / number ....

Marks

Question 4 **Begin a new booklet**

(a)

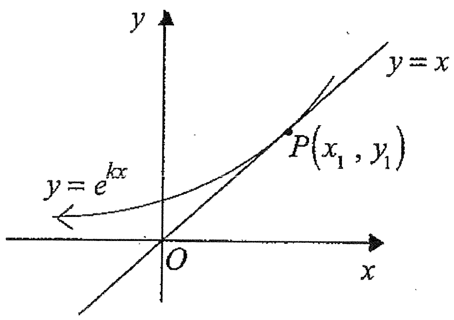


The region bounded by the curve $y = \frac{x}{x^2+1}$ and the x-axis between $x = 0$ and $x = 1$ is rotated through one complete revolution about the y-axis.

(i) Use the method of cylindrical shells to show that the volume V of the solid formed is given by $V = 2\pi \int_0^1 \frac{x^2}{x^2+1}\,dx$ **1**

(ii) Hence find the value of V in simplest exact form. **3**

(b)



The line $y = x$ is tangent to the curve $y = e^{kx}$ (where $k > 0$) at the point $P(x_1, y_1)$ on the curve. By considering the gradient of OP show that $k = \frac{1}{e}$. **3**

Question 4 continued

(c) The Hyperbola H has the equation $\frac{x^2}{25} - \frac{y^2}{9} = 1$.

(i) Find the eccentricity of H. **1**

(ii) Find the co-ordinates of the foci of H. **1**

(iii) Draw a neat one third of a page sketch of H. **2**

(iv) The line $x = 6$ cuts H at A and B. Find the coordinates of A and B if A is in the first quadrant. **2**

(v) Derive the equation of the tangent to H at A. **2**

Student name / number ....

Marks

Question 5 **Begin a new booklet**

(a) A lifebelt mould is made by rotating the circle $x^2 + y^2 = 64$
Through one complete revolution about the line $x = 28$, where
All the measurements are in centimetres.

(i) Use the method of slicing to show that the volume, $V\ cm^3$ **5**

of the lifebelt is given by

$V = 112\pi \int_{-8}^{8} \sqrt{64 - y^2\, dy}.$

(ii) Find the exact volume of the lifebelt. **2**

(b) (i) Show that the tangent to the ellipse $\dfrac{x^2}{12} + \dfrac{y^2}{4} = 1$ at the point **8**

P(3, 1) has equation $x + y = 4$.

ii) If this tangent cuts the directrix in the fourth quadrant at the point T,

and S is the corresponding focus, show that SP and ST are perpendicular.

Student name / number ....

Marks

Question 6 **Begin a new booklet**

(a) i) Show that $\tan(A + \frac{\pi}{2}) = -\cot A$. **2**

ii) Use the method of Mathematical Induction, and the result in (i), to show that **4**

$$\tan\left\{(2n+1)\frac{\pi}{4}\right\} = (-1)^n \qquad \text{for all integers } n \geq 1.$$

(b) Given the equation $y^2 + xy + x^2 = 1$

i) Make y the subject. **2**

ii) Hence, <u>or otherwise</u>, find $\frac{dy}{dx}$ **2**

(c) Given that $z = \cos\theta + i\sin\theta$ and $z^n + z^{-n} = 2\cos n\theta$, show that **4**

$$\cos^4\theta = \tfrac{1}{8}\cos 4\theta + \tfrac{1}{2}\cos 2\theta + \tfrac{3}{8}$$

(d) Show that $\cos\left(x + \frac{\pi}{2}\right) = -\sin x$. **1**

Student name / number ....

Marks

Question 7 **Begin a new booklet**

(a)(i) Show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. **2**

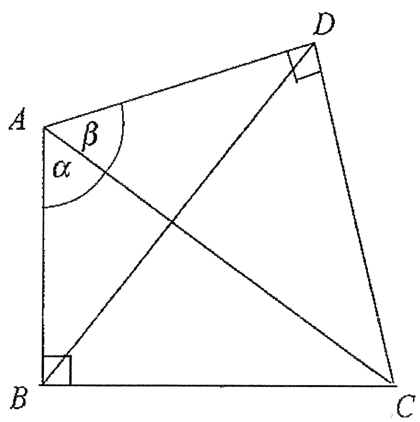
(ii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sqrt{1+(x-\frac{\pi}{4})^2}}\,dx$. **3**

(b) Let $I_n = \int_0^1 (1-x^r)^n\,dx$, where $r > 0$, for $n = 0, 1, 2, \ldots$.

(i) Show that $I_n = \dfrac{nr}{nr+1} I_{n-1}$ for $n = 1, 2, 3, \ldots$. **3**

(ii) Hence evaluate $\int_0^1 (1-x^{\frac{3}{2}})^3\,dx$. **2**

(c)



$ABCD$ is a quadrilateral in which $\angle ABC = \angle ADC = \frac{\pi}{2}$, $\angle CAB = \alpha$, $\angle CAD = \beta$ and $AC = 1$.

(i) Show that $\angle BDC = \alpha$. **2**

(ii) Hence show that $BD = \sin(\alpha+\beta)$. **3**

Student name / number ....

Marks

Question 8 **Begin a new booklet**

a) i) Write the general solution to $\tan 4\theta = 1$ **1**

ii) Use De Moivre's Theorem and the binomial theorem to show that $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$ **3**

iii) Hence find the roots of $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$ in the form $x = \tan\theta$. **3**

iv) Hence prove that : **2**

$$\tan^2\frac{\pi}{16} + \tan^2\frac{3\pi}{16} + \tan^2\frac{5\pi}{16} + \tan^2\frac{7\pi}{16} = 28$$

(Hint: Let the roots be α, β, γ and δ).

(b) (i) Use a diagram to explain why **1**

$$\int_0^b \sin x\, dx = \lim_{n\to\infty}\left(\sin\frac{b}{n} + \sin\frac{2b}{n} + \ldots + \sin\frac{nb}{n}\right)\cdot\frac{b}{n}$$

for $b = \dfrac{\pi}{2}$.

(ii) Given that $2\sin\theta\sin\alpha = \cos(\theta - \alpha) - \cos(\theta + \alpha)$, show that **2**

$$\sum_{k=1}^{n}\sin\left(\frac{kb}{n}\right) = \frac{\cos\left(\frac{b}{2n}\right) - \cos\left(b + \frac{b}{2n}\right)}{2\sin\left(\frac{b}{2n}\right)}$$

(iii) Hence show that $\int_0^b \sin x dx = 1 - \cos b$. **3**

END OF EXAM

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$