Name: ______________________________

Teacher: ______________________________

Class: ______________________________

FORT STREET HIGH SCHOOL

2011

HIGHER SCHOOL CERTIFICATE COURSE

ASSESSMENT TASK 3: TRIAL HSC

Mathematics Extension 2

TIME ALLOWED: 3 HOURS
(PLUS 5 MINUTES READING TIME)

Outcomes Assessed
Determines the important features of graphs of a wide variety of functions, including conic sections
Applies appropriate algebraic techniques to complex numbers and polynomials
Applies further techniques of integration, such as slicing and cylindrical shells, integration by parts and recurrence formulae, to problems
Uses ideas and techniques from calculus to solve problems in mechanics involving resolution of forces, resisted motion and circular motion
Synthesises mathematical solutions to harder problems and communicates them in an appropriate form

Question	1	2	3	4	5	6	7	8	Total	%
Marks	/15	/15	/15	/15	/15	/15	/15	/15	/120	

Directions to candidates:

- Attempt all questions
- The marks allocated for each question are indicated
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board – approved calculators may be used
- Each new question is to be started in a new booklet

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1: **Marks**

a) Find $\int \frac{e^{2x}-1}{e^{x}-1}dx$ **1**

b) Find $\int \frac{\tan x}{\tan 2x}dx$ **2**

c) Show that $\int_{2}^{4} \frac{dx}{x\sqrt{x-1}}dx = \frac{\pi}{6}$ **3**

d) Find $\int \frac{x^2+5x-4}{(x-1)(x^2+1)}dx$ **4**

e) The integral I_n is defined by $I_n = \int_{0}^{1} x^n e^{-x} dx$.

i. Show that $I_n = nI_{n-1} - e^{-1}$. **2**

ii. Hence show that $I_3 = 6 - 16e^{-1}$. **3**

Question 2: **Marks**

a) Given $z=\sqrt{3}-i$:

i. Express z in modulus-argument form. **2**

ii. Hence evaluate $\left(\sqrt{3}-i\right)^6$. **2**

b) $z=1+i$ is a root of the equation $z^2-aiz+b=0$, where a and b are real numbers.

i. Find the values of a and b. **2**

ii. Find the other root of the equation. **2**

c) The complex number z is given in modulus/argument form by $z=r\left(\cos\theta+i\sin\theta\right)$. Show that $\dfrac{z}{z^2+r^2}$ is real. **3**

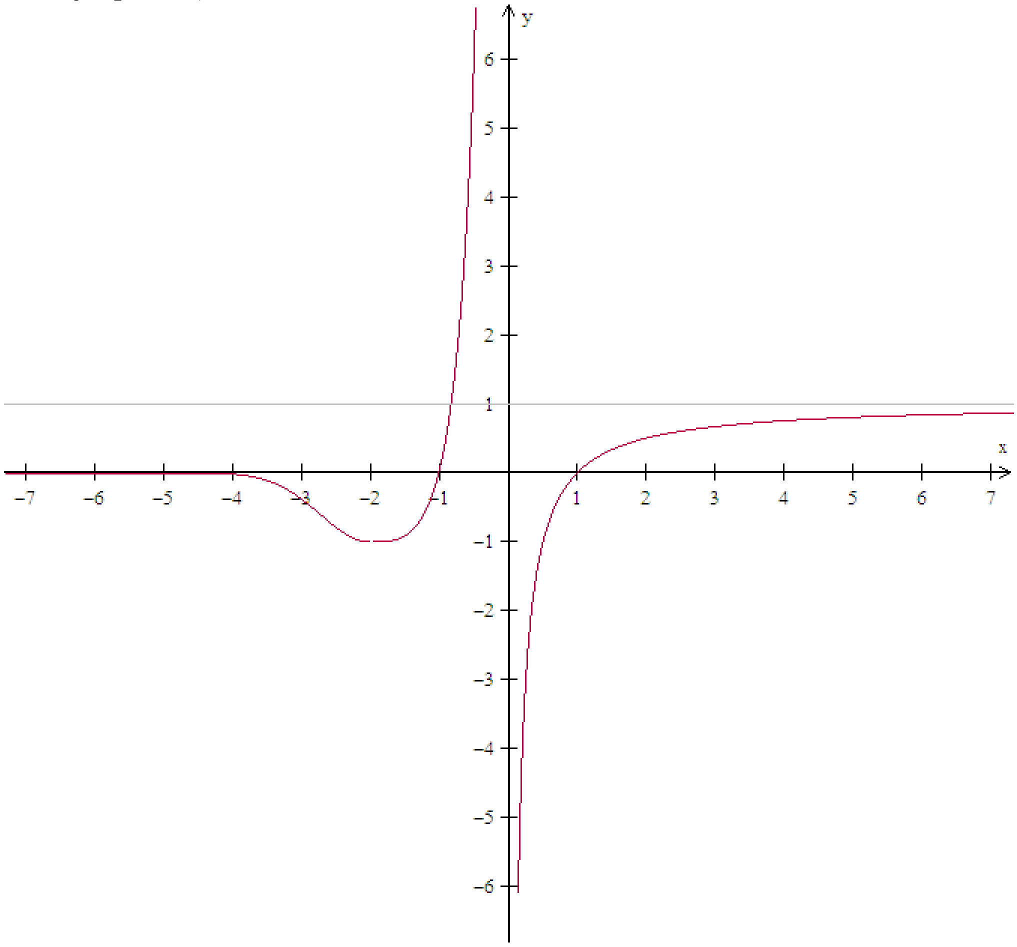
d) The locus of all points z in the complex plane which satisfy $\arg\left(\dfrac{z-3}{z+1}\right)=\dfrac{\pi}{3}$ forms part of a circle.

i. Sketch this locus. **2**

ii. Find the centre and radius of the circle. **2**

Question 3: **Marks**

a) The graph of $y = f(x)$ is shown below. (The lines $y = 1$ and the x-axis are asymptotes.)



Draw a neat one-third page sketch of the following, showing relevant features:

i. $y = |f(x)|$ **1**

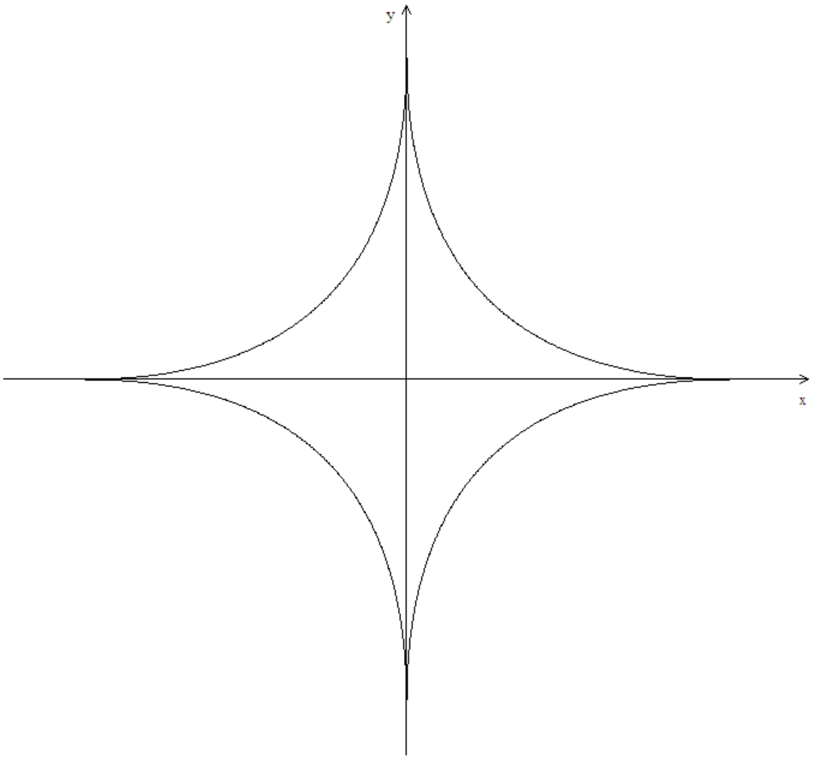
ii. $y = f(-x)$ **1**

iii. $y = \dfrac{1}{f(x)}$ **2**

iv. $y = (f(x))^2$ **2**

v. $y = e^{f(x)}$ **2**

b) The diagram shows the graph of the relation $|x|^{\frac{1}{2}}+|y|^{\frac{1}{2}}=L^{\frac{1}{2}}$, for $L>0$.



i. Show that the area of the region enclosed by the curve is $\frac{2}{3}L^2$ **3**

ii. A stone column has height H metres. Its base is the region enclosed by the curve $|x|^{\frac{1}{2}}+|y|^{\frac{1}{2}}=L^{\frac{1}{2}}$, and the cross section taken parallel to the base at height h metres is a similar region enclosed by the curve $|x|^{\frac{1}{2}}+|y|^{\frac{1}{2}}=l^{\frac{1}{2}}$ where $l=L\left(1-\frac{h}{H}\right)$. Find the volume of the stone column (in terms of *L* and *H*). **4**

Question 4: **Marks**

a) It is given that the hyperbola $xy = c^2$ touches (is tangential to) the parabola $y = x - x^2$ at point *Q* and crosses the parabola again at point *R*:

i. Show this information on a sketch **1**

ii. Deduce that the equation $x^3 - x^2 + c^2 = 0$ has a repeated root and hence find the value of c^2 **3**

iii. Find the coordinates of point *R*. **1**

b) $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

i. Show that the equation of the normal to the hyperbola at point P has the equation $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$ **4**

ii. The line through P parallel to the $y-$axis meets the asymptote $y = \frac{bx}{a}$ at Q. The tangent at P meets the same asymptote at R. The normal at P meets the $x-$axis at G. Prove that $\angle RQG$ is a right-angle. **2**

c) The region between the curve $y = \sin x$ and the line $y = 1$. From $x = 0$ to $x = \frac{\pi}{2}$ is rotated about the line $y = 1$. Using a slicing technique, find the exact volume of the solid thus formed. **4**

Question 5: **Marks**

a)

i. Find the general solution to the equation $\cos 4\theta = \frac{1}{2}$ **2**

ii. Use De Moivre's Theorem to show that
$\cos 4\theta = 8\cos^4 \theta - 8\cos^2 \theta + 1$ **3**

iii. Show that the equation $16x^4 - 16x^2 + 1 = 0$ has roots
$x_1 = \cos\frac{\pi}{12}, x_2 = \cos\frac{5\pi}{12}, x_3 = \cos\frac{7\pi}{12}, x_1 = \cos\frac{11\pi}{12}$ **2**

iv. By considering this equation as a quadratic in x^2, show that
$\cos\frac{\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}$. **3**

b) A particle is moving in the positive direction along a straight line in a medium that exerts a resistance to motion proportional to the cube of the velocity. No other forces act on the particle, i.e. $\ddot{x} = -kv^3$, where k is a positive constant.

At time $t = 0$, the particle is at the origin and has a velocity U. At time $t = T$, the particle is at $x = D$ and has a velocity $v = V$.

i. Using $\ddot{x} = \frac{dv}{dt}$, show that $\frac{1}{V^2} - \frac{1}{U^2} = 2kT$. **2**

ii. Using the identity $\ddot{x} = v\frac{dv}{dx}$, show that $\frac{1}{V} - \frac{1}{U} = kD$. **3**

Question 6: **Marks**

a) The equation $x^3 + kx + 2 = 0$ has roots α, β and γ.

i. Find an expression for $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ in terms of k. **2**

ii. Show that the value of $\alpha^3 + \beta^3 + \gamma^3$ is independent of k. **2**

iii. Find the monic equation with roots $\alpha^2, \beta^2, \gamma^2$ (leaving coefficients in terms of k). **3**

b) A particle of mass m falls under gravity in a medium whose resistance R to the motion is proportional to the square of the speed $\left(R = mkv^2\right)$. Acceleration due to gravity is g.

i. Find an expression for the terminal velocity V_t in this medium. **2**

A second particle of mass M is projected vertically upward from ground level in the same medium with an initial velocity U. It takes T seconds to reach its maximum height H above the projection point.

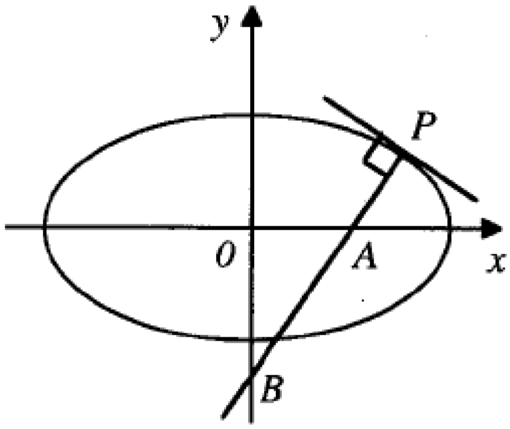
ii. Show that $T = \frac{V_t}{g} \tan^{-1}\left(\frac{U}{V_t}\right)$. **3**

iii. Show that $H = \frac{V_t^2}{2g}\left[\ln\left(\frac{V_t^2 + U^2}{V_t^2}\right)\right]$. **3**

Question 7: **Marks**

a) A solid has a base in the form of an ellipse with major axis 10 units and minor axis 8 units. Find the volume of the solid formed if every section perpendicular to the major axis is an isosceles triangle with altitude 6 units. **3**

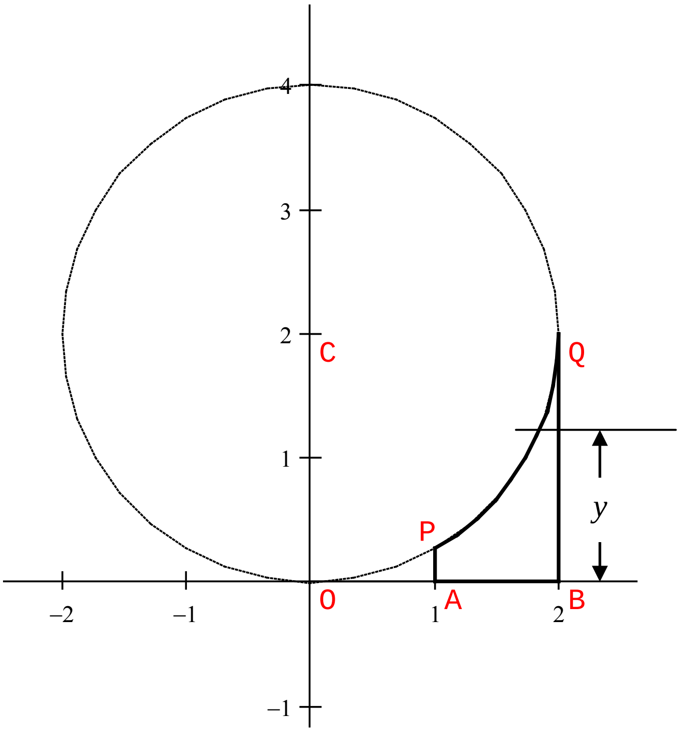
b) $P\left(a\cos\theta, b\sin\theta\right)$, where $0<\theta<\frac{\pi}{2}$, is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $0<b<a$. The normal to the ellipse at ***P*** has equation $ax\sin\theta - by\cos\theta = \left(a^2-b^2\right)\sin\theta\cos\theta$. This normal cuts the x-axis at ***A*** and the y-axis at ***B***.



i. Show that ΔOAB has an area given by $\frac{\left(a^2-b^2\right)^2}{2ab}\sin\theta\cos\theta$. **3**

ii. Find the maximum area of ΔOAB and the coordinates of ***P*** where this maximum occurs. **3**

c) In the diagram below, the shaded region is bounded by the lines $x = 1$, $x = 2$ the curve $x^2 + (y-2)^2 = 4$ and the x-axis. This region is to be rotated about the y-axis. When the region is rotated, the line segment bounded on the left by the curve at height y sweeps out an annulus.



i. Show that the area of the annulus at height y is given by $\pi(y-2)^2$, where $2-\sqrt{3} \le y \le 2$. **2**

ii. Hence find the exact volume of the solid if the entire region is rotated about the y-axis, given that the cylindrical pipe portion of the solid has a volume of $\pi\left(6-3\sqrt{3}\right)$. **2**

d) The complex number $\frac{\sqrt{3}}{2}+\frac{i}{2}$ is one of the n^{th} roots of -1. Find the least value of n for this to be so. **2**

Question 8: **Marks**

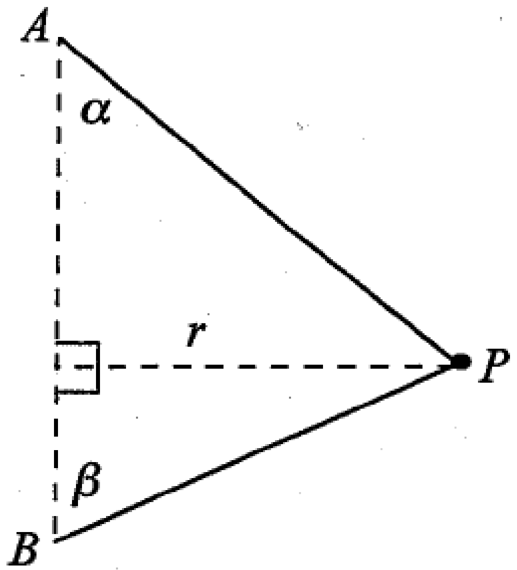
a) A straight line is drawn through a fixed point $P(a,b)$ in the first quadrant of the number plane. The line cuts the positive x-axis at ***A*** and the positive y-axis at ***B***. Given $\angle OAB = \theta$:

i. Prove that the length of ***AB*** is given by $AB = a\sec\theta + b\,\text{cosec}\,\theta$. **2**

ii. Show that the length ***AB*** will be a minimum if $\cot\theta = \left(\frac{a}{b}\right)^{\frac{1}{3}}$. **3**

iii. Show that the minimum length of ***AB*** is $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$. **2**

b) ***A*** and ***B*** are two fixed points with ***B*** vertically below ***A***. ***P*** is a particle with mass ***M*** kg. Two strings with ends fixed at ***A*** and ***B*** are fastened to ***P***. Particle ***P*** moves in a horizontal circle of radius ***r*** metres with a constant angular velocity of ω radians per second so that both strings remain taut, making angles of α, β respectively with the vertical. The tension in the strings ***AP*** and ***BP*** are T_1 Newtons and T_2 Newtons respectively. The acceleration due to gravity is ***g*** ms^{-2}.



i. Draw a diagram showing the forces acting on particle ***P***. **1**

ii. By resolving forces, show that $T_1\cos\alpha - T_2\cos\beta = Mg$ and $T_1\sin\alpha + T_2\sin\beta = Mr\omega^2$. **3**

iii. Hence show that $T_2 = \frac{M\left(r\omega^2\cos\alpha - g\sin\alpha\right)}{\sin\left(\alpha+\beta\right)}$. Find the corresponding expression for T_1. **2**

iv. Find the smallest possible value of ω for the motion to continue as described. Explain what happens if ω drops below this value. **2**