FORT STREET HIGH SCHOOL

Name:

Teacher: Please tick

- Mr Bayas
- Mr Fraser
- Mr Hayes

2010

HIGHER SCHOOL CERTIFICATE COURSE

ASSESSMENT TASK 4: TRIAL HSC

Mathematics Extension 2

TIME ALLOWED: 3 HOURS
(PLUS 5 MINUTES READING TIME)

Outcomes Assessed	Questions	Marks
Determines the important features of graphs of a wide variety of functions, including conic sections	2, 4	
Applies appropriate algebraic techniques to complex numbers and polynomials	1, 3	
Applies further techniques of integration, such as slicing and cylindrical shells, integration by parts and recurrence formulae, to problems	5, 6	
Synthesises mathematical solutions to harder problems and communicates them in an appropriate form, resisted motion	8,7	

Question	1	2	3	4	5	6	7	8	Total	%
Marks	/15	/15	/15	/15	/15	/15	/15	/15	/120	

Directions to candidates:

- Attempt all questions
- The marks allocated for each question are indicated
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board – approved calculators may be used
- Each new question is to be started in a new booklet

Question 1 (15 Marks) **Start a new booklet** **Marks**

a) Let $z = 5 - 6i$ and $w = 3 + 4i$. Express the following in the form a + ib where a and b are real numbers.

(i) z^2 1

(ii) $\frac{z}{w}$ 2

b) (i) Express $w = 8 + 8i$ in modulus-argument form 1

(ii) Hence, or otherwise find all numbers z such that $z^5 = 8 + 8i$ giving your answer in modulus-argument form. 3

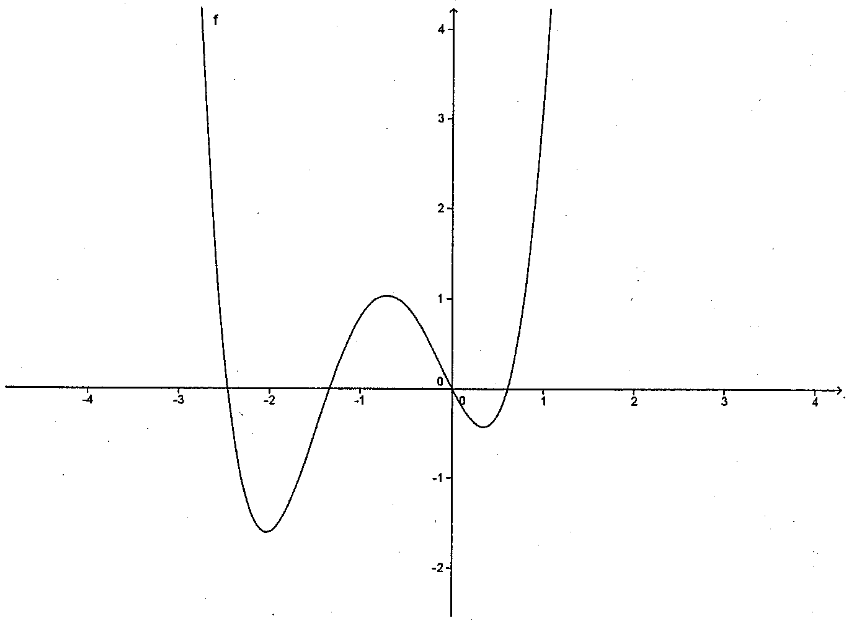
c) Sketch the region in the Argand diagram defined by $|z - 2 + i| < 3$ *and* $-\frac{\pi}{3} \leq \arg(z - 2 + i) \leq \frac{\pi}{3}$ 3
Indicate whether corner points are included or excluded. You do not need to find coordinates of the corner points or intercepts.

d) Find $\sqrt{1 + i}$ in the form a + ib where a and b are real numbers. Hence find an exact value for $\tan\left(\frac{\pi}{8}\right)$. 3

e) Given Euler's formula $e^{i\theta} = cos\theta + isin\theta$. The complex number z can be expressed in polar form as $z = re^{i\theta}$ where $r = |z|$ *and* $\theta = \arg(z)$. Use the polar form of z to find $\ln(z)$ and hence find $\ln(1 + i)$ in the form a + ib where a and b are real numbers. 2

Question 2 (15 Marks) **Start a new booklet** **Marks**

a) The diagram shows the graph of $y = f(x)$



Draw separate one third page sketches of the graphs of the following

(i) $y = \dfrac{1}{f(x)}$ 2

(ii) $y^2 = f(x)$ 2

(iii) $y = 2^{f(x)}$ 2

(iv) $y = f(\frac{1}{x})$ 2

Question 2 continued **Marks**

b) Consider the function $f(x) = \ln(2 + 2\,cos(2x))$, $-2\pi \le x \le 2\pi$

(i) Show that the function f is even and the curve $y = f(x)$ is concave down for all values of x in its domain, except where its not defined. 3

(ii) Sketch using a third of a page, the graph of the curve $= f(x)$. 2

c) Find the coordinates of the points where the tangent to the curve $x^2 + 2xy + 3y^2 = 18$ is horizontal. 2

End of Question 2

Next question, Question 3 on the next page , page 4

Question 3 (15 Marks) **Start a new booklet** **Marks**

a) (i) Prove the theorem — 2
If α is a zero of multiplicity r of the real polynomial equation $P(x) = 0$, then α is a zero of multiplicity $r - 1$ of $P'(x) = 0$.

(ii) The polynomial equation $3x^5 - ax^2 + b = 0$ has a multiple root. — 3
Show that $8788a^5 = 28125b^3$

b) The polynomial $P(z)$ is defined by — 3

$$P(z) = z^4 - 2z^3 - z^2 + 2z + 10$$

Given that $z - 2 + i$ is a factor of $P(z)$, express $P(z)$ as a product of real quadratic factors.

c) (i) Show that $\cos(P+Q) + \cos(P-Q) = 2\cos P\cos Q$. — 1

(ii) Let α and β be the roots of the equation $z^2 \sin^2\emptyset - z\sin 2\emptyset + 1 = 0$.

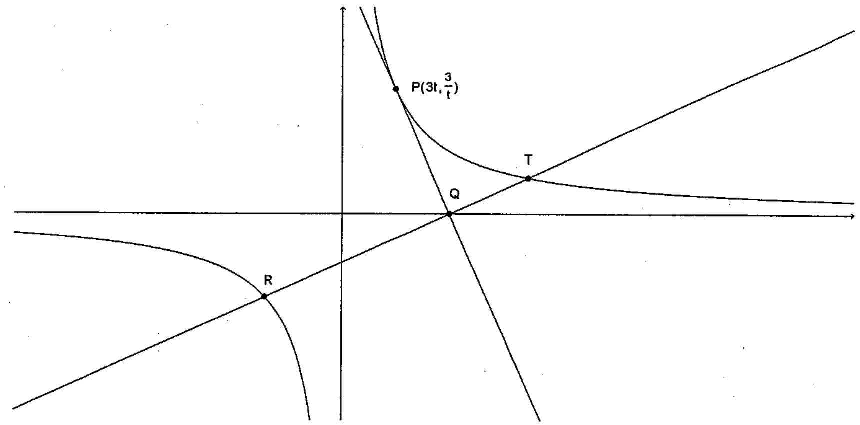
1. Show that $\alpha + \beta = 2\cos\emptyset\, cosec\emptyset$ — 1
2. Show that $\alpha^2 + \beta^2 = 2\cos 2\emptyset\, cosec^2\emptyset$ — 1
3. Hence by mathematical induction, — 4

prove that if n is a positive integer then

$\alpha^n + \beta^n = 2\cos n\emptyset\, cosec^n\emptyset$

Question 4 (15 Marks) **Start a new booklet** **Marks**

a) $P(3t, \frac{3}{t})$ is a point on the rectangular hyperbola xy = 9. The tangent at P cuts the x axis at Q and the line through Q, perpendicular to the tangent at P, cuts the hyperbola at the points R and T as shown



(i) Show that the equation of the tangent at P is $x + t^2y = 6t$. — 2

(ii) Show that the line through Q, perpendicular to the tangent at P, has equation $t^2x - y = 6t^3$ — 3

(iii) If M is the midpoint of RT, show M has coordinates $(3t, -3t^3)$. — 3

(iv) Find the equation of the locus of M, as P moves on the curve xy = 9 . — 1

b) The Hyperbola H has equation $x^2 - 3y^2 = 6$
Show that the equation of the normal to H at $P(2\sqrt{2}, \sqrt{2})$ is $3x + 2y = 8\sqrt{2}$. — 2

c) The Points $M(a\cos\propto, b\sin\propto)$ and $N(-a\sin\propto, b\cos\propto)$ lie on the ellipse E: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Find the equations of the tangents at M and N and show these tangents intersect at the point $P(\,a(\cos\propto - \sin\propto), b(\sin\propto + \cos\propto))$. — 4

Question 5 (15 Marks) **Start a new booklet** **Marks**

a) Evaluate correct to 3 decimal places

$$\int_0^1 \frac{e^{2x}dx}{e^{4x}+1}$$ 2

b) Find $\int \frac{dp}{\sqrt{9+8p-p^2}}$ 2

c) Using the substitution $t = tan\frac{\theta}{2}$, find

$$\int \frac{2d\theta}{5-4sin\theta}$$ 3

d) Find $\int \frac{x^5-7x^2+8}{x^3-8}\, dx$ 4

e) If $I_n = \int_0^{\frac{\pi}{4}} sec^n x dx$ for n ≥ 0 4

(integral from zero to pi over 4 of secx to the power n dx)

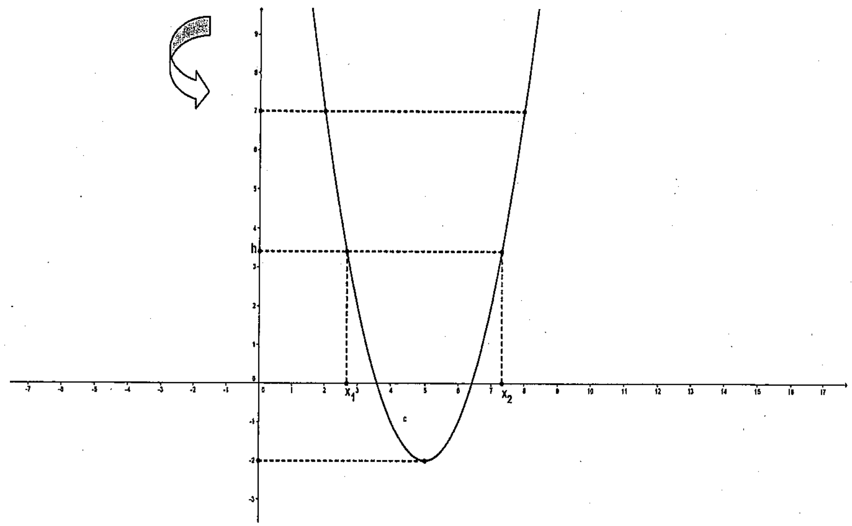
show that

$$I_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2} \quad \text{for } n \ge 2$$

and deduce $I_6 = \frac{28}{15}$

Question 6 (15 Marks) **Start a new booklet** **Marks**

a) A flat top parabolic torus is formed by rotating the area inside the parabola $y = x^2 - 10x + 23$ between the lines $y = -2$ *and* $y = 7$ around the y axis.



The cross section at $y = h$ *where* $-2 \le h \le 7$, is an annulus. The annulus has inner radius x_1 and outer radius x_2 where x_1 *and* x_2 are the solutions to $x^2 - 10x + 23 = h$

(i) Find x_1 and x_2 in terms of h 1

(ii) Find the area of the cross-section at height h, in terms of h. 2

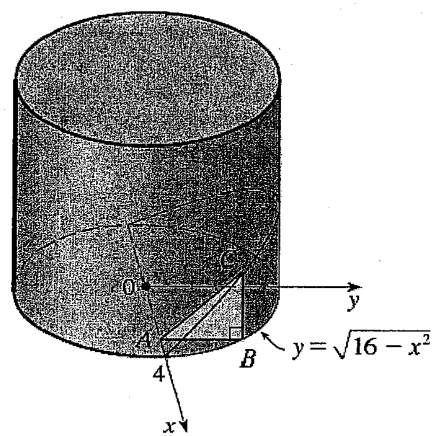
(iii) Find the volume of the flat top parabolic torus. Leave answer in exact form. 2

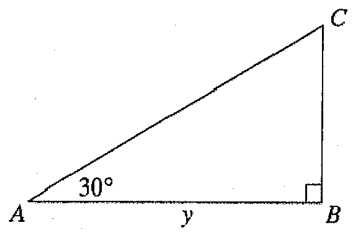
Question 6. Continued **Marks**

b) A wedge is cut out of a circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder.
The other intersects the first at an angle of 30 along a diameter of the cylinder.

(i) Show the cross sectional area is $A(x) = \frac{16-x^2}{2\sqrt{3}}$ 2

(ii) Hence find the volume of the wedge. 3

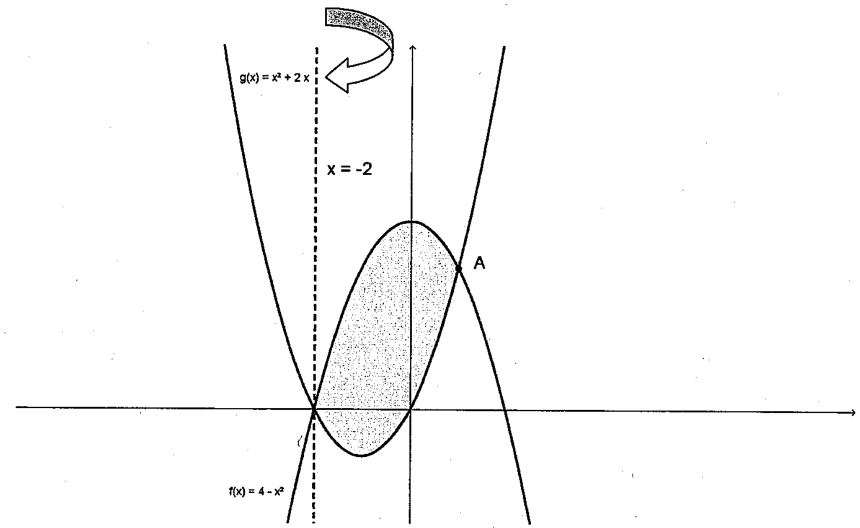




Question 6. continued **Marks**

c)

The lightly shaded region bounded by $y = 4 - x^2, y = x^2 + 2x$ is rotated about the line $x = -2$. The point A is the intersection of $y = 4 - x^2$ *and* $y = x^2 + 2x$ in the first quadrant.



(i) Find the coordinate of A 1

(ii) Use the method of cylindrical shells to express the volume of the resulting solid of revolution as an integral. 2

(iii) Evaluate the integral in part (ii), leave answer in exact form. 2

Question 7. (15 Marks) **Start a new booklet** — Marks

a) A cannon ball of mass 1 kilogram is projected vertically upward from the origin with an initial speed of $20m/s$. The cannon ball is subjected to gravity $10ms^{-2}$ and air resistance $\frac{v^2}{20}$.
The upward equation of motion is
$$\ddot{y} = -\frac{v^2}{20} - 10$$

(i) Using $\ddot{y} = v\frac{dv}{dy}$ show that while the cannon ball Is rising $v^2 = 600e^{-\frac{y}{10}} - 200$ — 3

(ii) Hence find the maximum height reached by the cannon ball correct to 2 decimal places. — 1

(iii) Using $\ddot{y} = \frac{dv}{dt}$ find how long the cannon ball takes to reach this maximum height correct to 2 decimal places? — 2

(iv) How fast is the cannon ball travelling when it returns to the origin correct to 2 decimal places? — 3

b) A cylindrical water tank has a constant cross-sectional area A.
Water drains through a hole at the bottom of the tank.
The Volume of water decreases at a rate (− k times the cube root of h)
$\frac{dV}{dt} = -k\sqrt[3]{h}$ Where k is a positive constant and h is the depth of water.
Initially the tank is full and it takes T seconds to drain. Thus $h = h_0$ $when$ $t = 0$
And $h = 0$ $when$ $t = T$.

(i) Show that $\frac{dh}{dt} = -\frac{k}{A}\sqrt[3]{h}$ — 2

(ii) By considering the equation for $\frac{dt}{dh}$ or otherwise
Show $h^2 = {h_0}^2\left(1 - \frac{t}{T}\right)^3$. — 3

(iii) Suppose it takes 12 seconds for half the water to drain. How long does it take to empty the full tank? to nearest second. — 1

Question 8. (15 Marks) **Start a new booklet** — Marks

a) Let α be a real number and suppose z is a complex number such that
$$z + \frac{1}{z} = 2\cos\alpha$$

(i) By reducing the above equation to a quadratic equation in z, solve for z and use de Moivre's theorem to show that
$$z^n + \frac{1}{z^n} = 2\cos n\alpha .$$ — 3

(ii) Let $w = z + \frac{1}{z}$. Prove that
$$w^3 + w^2 - 2w - 2 = z + \frac{1}{z} + \left(z^2 + \frac{1}{z^2}\right) + \left(z^3 + \frac{1}{z^3}\right).$$ — 2

(iii) Hence, or otherwise , find all solutions of $\cos\alpha + \cos 2\alpha + \cos 3\alpha = 0,$ in the range $0 \le \alpha \le 2\pi$. — 3

b) Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, — 1

Hence evaluate $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx$. — 3

c) The area A of the surface of revolution generated by rotating a smooth arc $y = f(x), a \le x \le b$ around the x axis, is given by the integral formula — 3
$$A = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx$$

Rotate the circle $x^2 + y^2 = r^2$ around the x axis and show that the surface Area of the sphere generated is $4\pi r^2$.

End of Examination