Name: ______________________________

Teacher: ____________________________

Class: ______________________________

FORT STREET HIGH SCHOOL

2006

HIGHER SCHOOL CERTIFICATE COURSE

ASSESSMENT TASK 4: TRIAL HSC

Mathematics Extension 2

TIME ALLOWED: 3 HOURS
(PLUS 5 MINUTES READING TIME)

Outcomes Assessed	Questions	Marks
Determines the important features of graphs of a wide variety of functions, including conic sections	4, 6	
Applies appropriate algebraic techniques to complex numbers and polynomials	2, 3	
Applies further techniques of integration, such as slicing and cylindrical shells, integration by parts and recurrence formulae, to problems	1, 5	
Synthesises mathematical solutions to harder problems and communicates them in appropriate form	7, 8	

Question	1	2	3	4	5	6	7	8	Total	%
Marks	/15	/15	/15	/15	/15	/15	/15	/15	/120	

Directions to candidates:

- Attempt all questions
- The marks allocated for each question are indicated
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board – approved calculators may be used
- Each new question is to be started on a new page

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1: (15 marks)

a) Find $\int \frac{dx}{x^2 - 4x + 9}$ **2**

b)

i. Express $\frac{4x-2}{(x^2-1)(x-2)}$ in the form $\frac{Ax+B}{x^2-1} + \frac{C}{x-2}$,

where A, B and C are constants. **3**

ii. Hence evaluate

$$\int_3^6 \frac{4x-2}{(x^2-1)(x-2)} dx$$ **2**

c) Find $\int \frac{e^{2x}}{e^x - 1} dx$ by using the substitution $u = e^x$. **3**

d) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx$, where n is a non-negative integer.

i. Show that $I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \quad \cos^2 x dx$, where $n \geq 2$. **2**

ii. Deduce that $I_n = \left(\frac{n-1}{n}\right) I_{n-2}$ when $n \geq 2$. **2**

iii. Evaluate I_4 **1**

Question 2: (15 marks)

a) Let $z = \sqrt{3} + i$

i. Express z in modulus/argument form. **1**

ii. Show that $z^7 + 64z = 0$ **2**

b) Find the complex number $z = a + ib$, where a and b are real, such that

$$\text{Im}(z) + \bar{z} = \frac{1}{1+i}$$ **3**

c) The complex number z satisfies the condition $|z - 8| = 2\,\text{Re}(z - 2)$

i. Sketch the locus defined by this equation on an Argand diagram, showing any important features of the curve. State the type of curve and write down its equation. **3**

ii. Write down the value of $|z + 8| - |z - 8|$ **1**

iii. Find the possible values of $\arg z$ **2**

d) P, Q represent complex numbers α, β respectively in an Argand diagram, where O is the origin and O, P and Q are not collinear. In ΔOPQ, the median from O to the midpoint M of PQ meets the median from Q to the midpoint N of OP in the point R, where R represents the complex number z.

i. Show this information on a sketch **1**

ii. Explain why there are positive real numbers k, L so that

$$kz = \frac{1}{2}(\alpha + \beta) \text{ and } L(z - \beta) = \frac{1}{2}\alpha - \beta$$ **2**

Question 3: (15 marks)

a) The equation $x^3 + bx^2 + x + 2 = 0$ where b is a real number has roots α, β, γ

i. Obtain an expression in terms of b for $\alpha^2 + \beta^2 + \gamma^2$ **2**

ii. Hence determine the set of possible values of b if the roots of the above equation are real. **1**

iii. Write down the equation whose roots are $2\alpha,\ 2\beta,\ 2\gamma$. **2**

b)

i. Show that if a is a multiple root of the polynomial equation $f(x) = 0$ then $f(a) = f'(a) = 0$. **2**

ii. The polynomial $\alpha x^{n+1} + \beta x^n + 1$ is divisible by $(x-1)^2$.

Show that $\alpha = n$ and $\beta = -(1+n)$. **4**

iii. Prove that $1 + x + \dfrac{x^2}{2!} + \cdots\cdots + \dfrac{x^n}{n!}$ has no multiple roots for $n \geq 1$. **4**

Question 4: (15 marks)

The functions $S(x)$ and $C(x)$ are defined by the formulae

$S(x)=\frac{1}{2}\left(e^{x}-e^{-x}\right)$ and $C(x)=\frac{1}{2}\left(e^{x}+e^{-x}\right)$

a)

i. Verify that $S'(x)=C(x)$ **1**

ii. Show that $S(x)$ is an increasing function for all real x. **2**

iii. Prove that $\{C(x)\}^2=1+\{S(x)\}^2$. **2**

b)

i. $S(x)$ has an inverse function, $S^{-1}(x)$ for all values of x. Briefly justify this statement **2**

ii. Let $y=S^{-1}(x)$. Prove that $\frac{dy}{dx}=\frac{1}{\sqrt{1+x^2}}$ **4**

iii. Hence, or otherwise, show that

$$S^{-1}(x)=\ln\left\{x+\sqrt{1+x^2}\right\}$$ **1**

iv. Show that $\int_0^1 \frac{dx}{\sqrt{x^2+2x+2}}=\ln\left\{\frac{2+\sqrt{5}}{1+\sqrt{2}}\right\}$ **3**

Question 5: (15 marks)

a) Using the method of cylindrical shells find the volume of the solid formed when the region bounded by the curve $y = x^2 + 1$ and the x-axis between $x = 0$ and $x = 2$ is rotated about the y axis. **4**

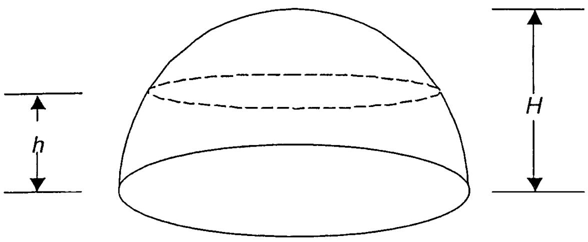
b)

i. Using the substitution $x = a\sin\theta$, or otherwise, verify that

$$\int_0^a \left(a^2 - x^2\right)^{\frac{1}{2}} dx = \frac{1}{4}\pi a^2$$ **4**

ii. Deduce that the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab. **3**

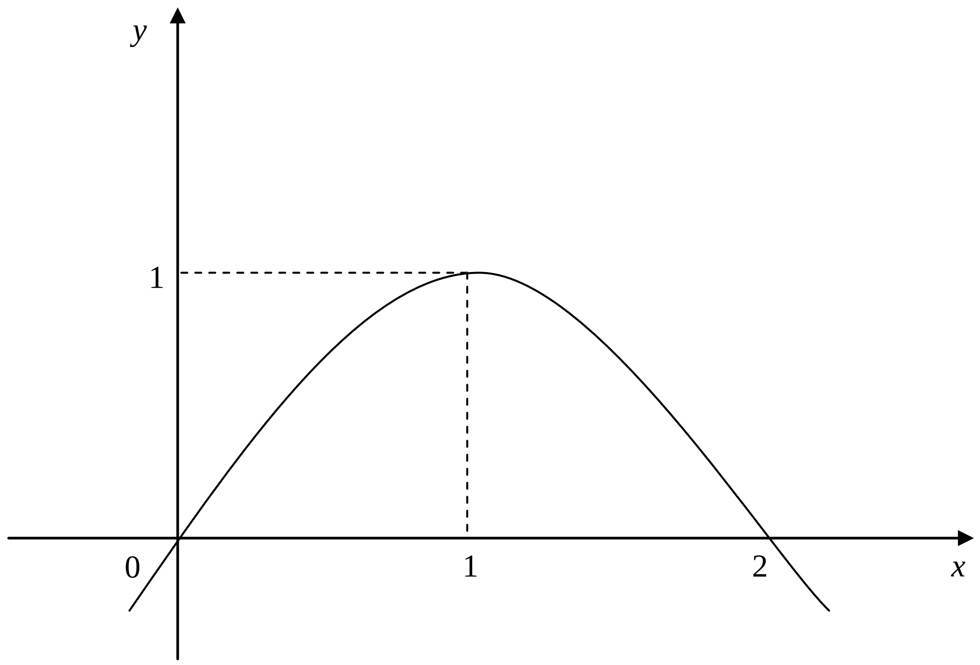
iii. The diagram below shows a mound of height H. At height h above the horizontal base, the horizontal cross-section of the mound is elliptical in shape with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \lambda^2$ where $\lambda = 1 - \frac{h^2}{H^2}$ and x and y are appropriate coordinates in the plane of the cross-section.



Show that the volume of the mound is $\frac{8\pi abH}{15}$. **4**

Question 6: (15 marks)

a) The graph below shows the curve $y = f(x)$ where $f(x) = x(2-x)$.



Without the use of calculus, sketch the following curves. Show any intercepts, asymptotes, end points and turning points.

i. $y = f(2x)$ **1**

ii. $y = \dfrac{1}{f(x)}$ **2**

iii. $|y| = f(x)$ **2**

iv. $y = \ln f(x)$ **2**

v. $y = f(e^x)$ **2**

(continued over)

b) Consider the function $f(x) = |1+x| + |1-x|$.

i. Show that $f(x)$ is an even function. **1**

ii. Sketch the graph of $y = f(x)$ clearly showing essential features. **2**

iii. Use the graph to find the set of values of the real number k for which $f(x) = k$ has exactly 2 real solutions. **1**

c) On separate diagrams sketch the graphs of the following curves, showing the equations of any asymptotes

i. $y = \left(\tan^{-1} x\right)^2$ **1**

ii. $y^2 = \tan^{-1} x$ **1**

Question 7: (15 marks)

a) The ellipse E has equation $\frac{x^2}{100}+\frac{y^2}{75}=1$

i. Sketch the curve E, showing on your diagram the coordinatesof the foci and the equation of each directrix. **2**

ii. Find the equation of the normal to the ellipse at the point $P(5,7.5)$. **2**

iii. Find the equation of the circle that is tangential to the ellipse at P and Q $(5,-7.5)$ **3**

b) The hyperbola H has equation $xy=4$

i. Sketch H and indicate on your diagram the positions and coordinates of all points at which H intersects the axes of symmetry. **1**

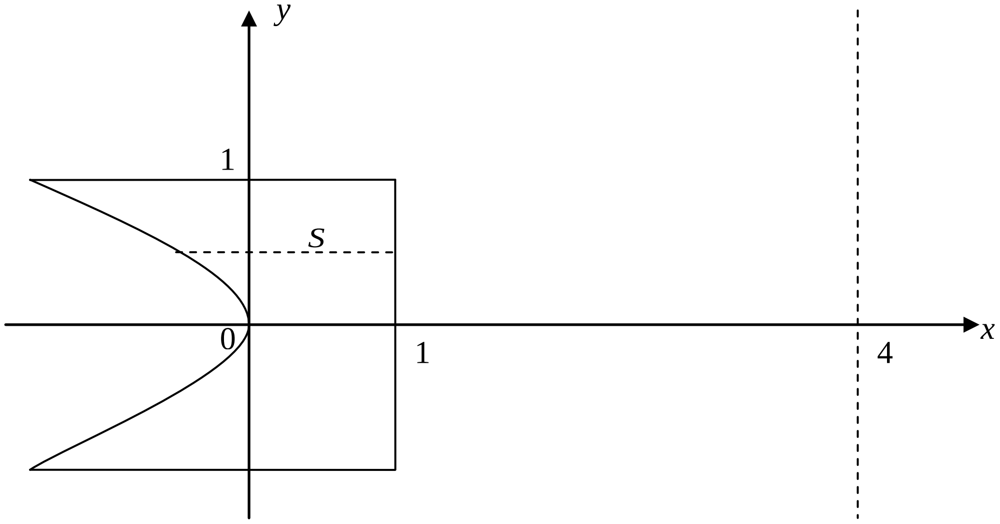
ii. Show that the equation of the tangent to H at $P\left(2t,\frac{2}{t}\right)$ where $t\neq 0$, is $x+t^2y=4t$. **2**

iii. If $s\neq 0$ and $s^2\neq t^2$, show that the tangents to H at P and $Q\left(2s,\frac{2}{s}\right)$ intersect at $M\left(\frac{4st}{s+t},\frac{4}{s+t}\right)$ **2**

iv. Suppose that in (iii) the parameter $s=-\frac{1}{t}$. Show that the locus of M is a straight line through, but excluding the origin. **3**

Question 8: (15 marks)

a)



The region bounded by the lines $x = 1$, $y = 1$, and $y = -1$ and by the curve $x + y^2 = 0$ is rotated through 360° about the line $x = 4$ to form a solid. As the region is rotated, the line segment S sweeps out an annulus.

i. Show that the area of the annulus swept by S at height y is equal to $\pi\left(y^4 + 8y^2 + 7\right)$ **3**

ii. Hence find the volume of the solid. **2**

b) The point $P(x_1, y_1)$ lies on the hyperbola of equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

i. Find the equation of the normal at P. **3**

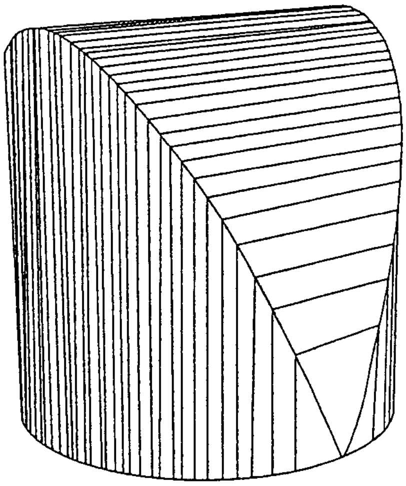
ii. The normal at P meets the x axis at G, and N is the foot of the perpendicular from P to the x axis. Show that

$$NG : ON = b^2 : a^2$$

where O is the origin. **2**

c) Trieu took a wooden cylinder and carved it into the shape shown in the diagram below. The base of her shape is a circle with radius 8 cm.

Each vertical cross-section shown in the diagram is a square.



i. Show that the area A of the cross-section distance x cm from the centre of the base is $A = 4\left(64 - x^2\right)$. **3**

ii. Hence show that the volume V of the art project is given by

$$V = 8\int_0^8 \left(64 - x^2\right)dx$$

and evaluate the integral. **2**

<u>END OF EXAMINATION</u>