Question One: (15 Marks) *Start a new sheet of paper.*

a) Find $\int \cos^3 x dx$. [2]

b) Find $\int \frac{dx}{\sqrt{x^2+2x+5}}$. [2]

c) Evaluate $\int_1^2 x \ln x dx$ (in exact form). [3]

d)

i) Find real numbers a, b and c such that
$\frac{1}{(x^2+1)(x+1)} = \frac{ax+b}{(x^2+1)} + \frac{c}{(x+1)}$. [2]

ii) Hence evaluate $\int_0^1 \frac{1}{(x^2+1)(x+1)} dx$ (in exact form). [2]

e) Use the substitution $x = \sin^2 \theta$ to evaluate $\int_0^{\frac{1}{2}} \frac{\sqrt{x}}{(1-x)^{\frac{3}{2}}} dx$ [4]

Question Two: (15 Marks) *Start a new sheet of paper.*

a) Given that $z = \sqrt{3} + \frac{1+i}{1-i}$ find:

i) $\operatorname{Im}(z)$ [1]

ii) $\overline{z}$ [1]

iii) z in mod/arg form. [2]

b) Solve $z^2 = 3-4i$. [2]

c) Illustrate on an Argand diagram the region given by
$\left\{ z : 0 \le \arg(z+4+i) \le \frac{2\pi}{3} \; and \; |z+4+i| \le 4 \right\}$. [3]

(Question 2 continued over)

d) z is a point on the circle $|z-1|=1$ and $\arg(z)=\theta$.

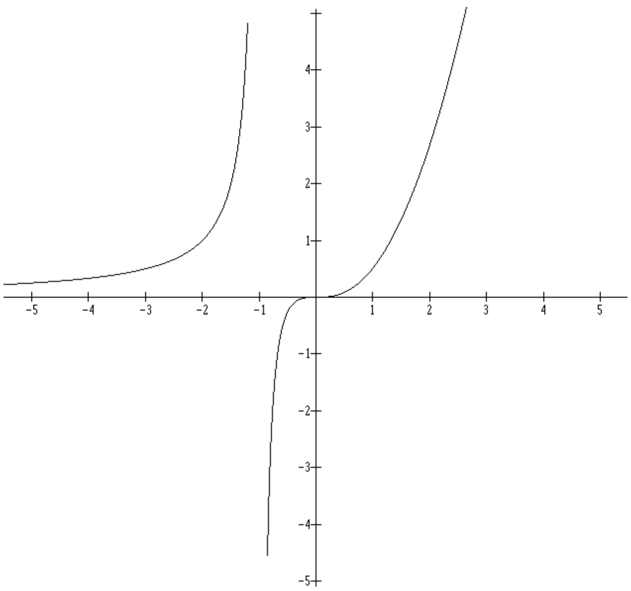
i) Find $\arg(z-1)$ in terms of θ. [1]

ii) Hence find $\arg(z^2-3z+2)$ in terms of θ. [2]

e) Find the complex fifth root of $-i$, in mod/arg form, and show these roots on an Argand diagram. [3]

<u>Question Three:</u> (15 Marks) *Start a new sheet of paper.*

a) The diagram shows the graph of $y=F(x)$. Draw neat sketches of (each should take about one third of a page):



i) $y=\dfrac{1}{F(x)}$ [2]

ii) $y=F(x)-|F(x)|$ [2]

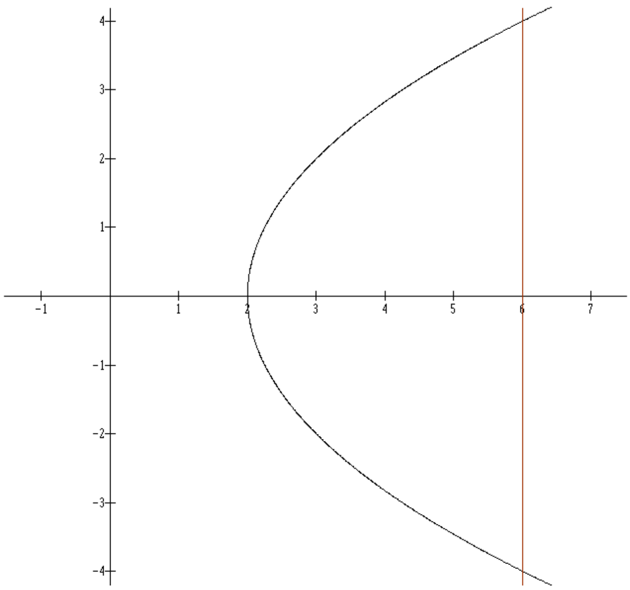
iii) $y=x.F(x)$ [2]

iv) $y=e^{F(x)}$ [2]

v) $y=\sqrt{F(x)}$ [2]

(Question 3 continued over)

b) The diagram shows the region bounded by the curve $y^2 = 4(x-2)$ and the line $x = 6$. Use the method of cylindrical shells to find the volume of the solid formed by rotating the given region about the $y-axis$. [5]



Question Four: (15 Marks) *Start a new sheet of paper.*

a)

i) Derive the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $P(a\cos\theta, b\sin\theta)$. [2]

ii) If $P(a\cos\theta, b\sin\theta)$ is on the ellipse in the first quadrant, and the tangent at P meets the $x-axis$ and the $y-axis$ at X and Y respectively, find the coordinates of X and Y. [2]

iii) For the triangle thus formed by OXY, find the minimum area of this triangle, and the coordinates of P (in terms of a and b) for this case. [5]

b)

i) Given $I_n = \int_0^{\frac{\pi}{4}} \tan^n x\,dx$ where n is a positive integer and $n \geq 2$,

show that $I_n = \frac{1}{n-1} - I_{n-2}$. [4]

ii) Hence evaluate $I_5 = \int_0^{\frac{\pi}{4}} \tan^5 x\,dx$. [2]

Question Five: (15 Marks) *Start a new sheet of paper.*

a) Factorise $Q(x) = x^6 - 3x^2 + 2$ over the complex number field, given that it has two double roots. [3]

b) The equation $x^3 + px + 1 = 0$ has three real non-zero roots α, β and δ.

i) Find the values of $\alpha^2 + \beta^2 + \delta^2$ and $\alpha^4 + \beta^4 + \delta^4$ in terms of p, and show that p must be strictly negative. [4]

ii) Find the monic equation, with co-efficients in terms of p, whose roots are $\frac{\alpha}{\beta\delta}, \frac{\beta}{\alpha\delta}, \frac{\delta}{\alpha\beta}$. [4]

c) Let z_1, z_2 and z_3 be three complex numbers represented by the points Z_1, Z_2 and Z_3 respectively on the Argand diagram, where $z_1 \times z_3 = (z_2)^2$. Show that OZ_2 bisects $\angle Z_1OZ_3$. [4]

Question Six: (15 Marks) *Start a new sheet of paper.*

a) PQRS is a cyclic quadrilateral. The bisector of $\angle$ PQS cuts the segment PR at X and the circle at M, and RM cuts the segment QS at Y.

i) Draw a neat diagram showing the above information. [1]

ii) Prove XQRY is a cyclic quadrilateral. [3]

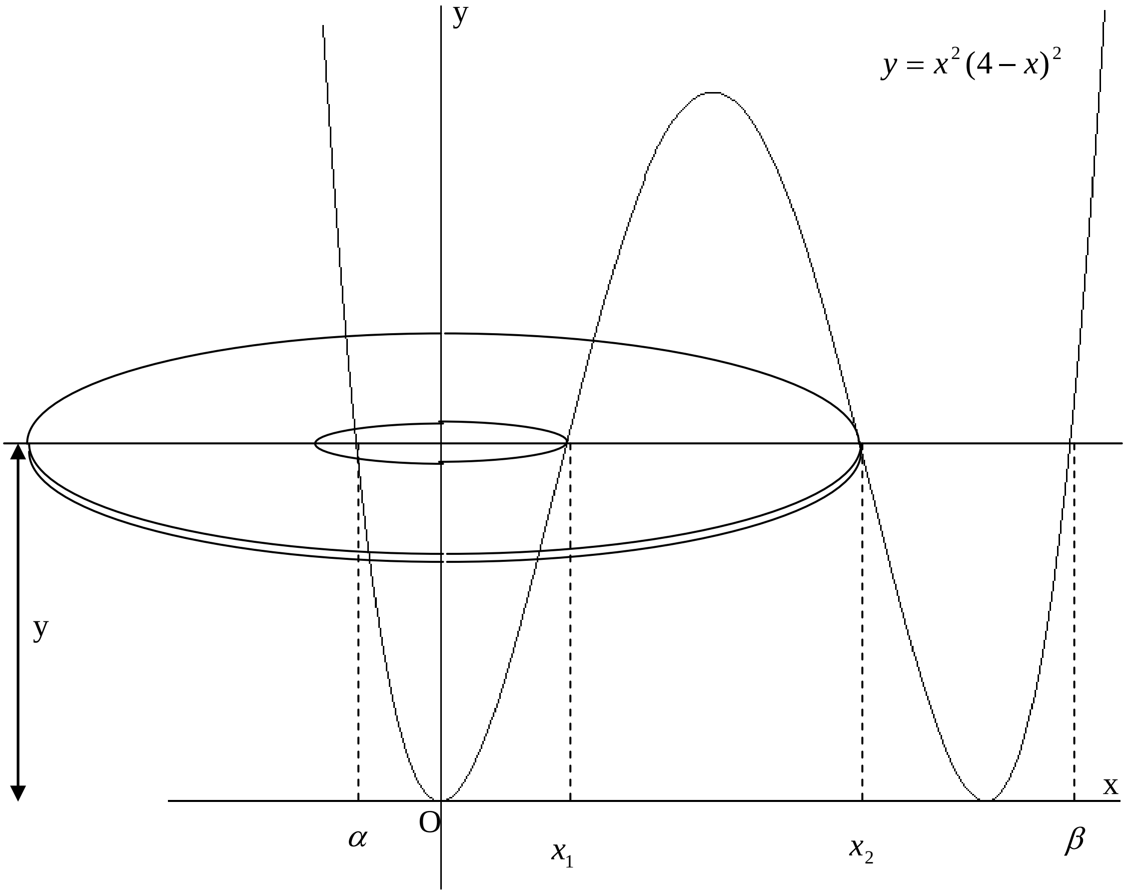
iii) Prove XY is parallel to PS. [3]

b) Find the limiting sum of the series $\frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + ... + \frac{n}{5^n} + ...$ [3]

c) From DeMoivre's Theorem, we know $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$. Use this to solve the equation $16x^4 - 16x^2 + 1 = 0$, and deduce the exact values of $\cos\frac{\pi}{12}$ and $\cos\frac{5\pi}{12}$. [5]

Question Seven: (15 Marks) *Start a new sheet of paper.*

a)



The region between $x = 0$ and $x = 4$ is rotated about the $y - axis$. The volume of the solid formed is found by taking slices perpendicular the the $y - axis$. The typical slice shown in the diagram is at a height y above the $x - axis$.

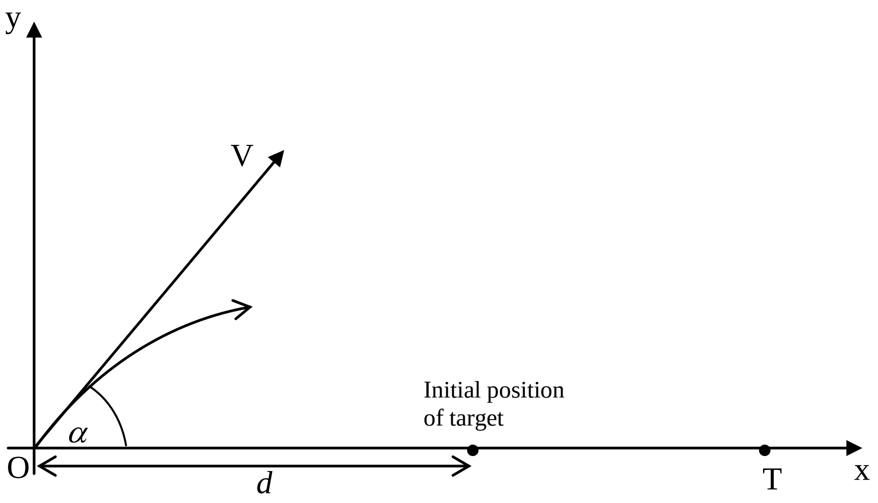
i) Deduce that α, x_1, x_2 and β, as shown in the diagram, are the roots of $x^4 - 8x^3 + 16x^2 - y = 0$. [1]

ii) Use the symmetry in the graph to explain why $\frac{x_1 + x_2}{2} = 2$ and $\frac{\alpha + \beta}{2} = 2$. Hence, by considering the co-efficients of the equation in (i), show that $\alpha\beta = -x_1 x_2$, and deduce that $x_1 x_2 = \sqrt{y}$ and that $x_2 - x_1 = 2\sqrt{4 - \sqrt{y}}$. [5]

(Question 8(a) continued over)

iii) Show that the volume of the solid of revolution is given by $V = 8\pi\int_0^{16}\sqrt{4-\sqrt{y}}\,dy$. Use the substitution $y=(4-u)^2$ to evaluate this integral and find the exact volume. [4]

b)



A projectile, of initial speed V m/s, is fired at an angle α from the origin O towards a target T, which is moving away from O along the $x-axis$.

You may assume that the projectile's trajectory is defined by the equations:

$x = Vt\cos\alpha$ and $y = Vt\sin\alpha - \frac{1}{2}gt^2$, where x and y are the horizontal and vertical displacements of the projectile in meters at time t seconds after firing, and where g is the acceleration due to gravity.

i) Show that the projectile is above the $x-axis$ for a total of $\frac{2V\sin\alpha}{g}$ seconds. [1]

ii) Show that the horizontal range of the projectile is $\frac{2V^2\sin\alpha\cos\alpha}{g}$ meters. [1]

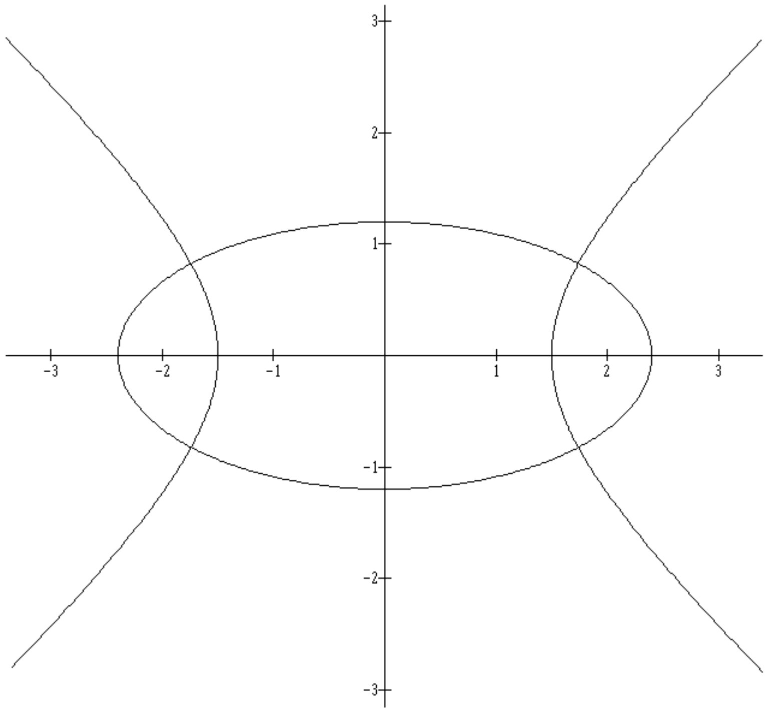
iii) At the instant the projectile is fired, the target T is d meters from O and is moving away at a constant speed of u m/s.

Suppose that the projectile hits the target when fired at an angle of elevation α. Show that $u = V\cos\alpha - \frac{gd}{2V\sin\alpha}$. [3]

Question Eight: (15 Marks) *Start a new sheet of paper.*

a) Find the volume of the solid generated by rotating the region common to the circles $x^2 + y^2 = 16$ and $x^2 + y^2 = 8x$ about their common chord. [8]

b) Hyperbola H has equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and eccentricity e. Ellipse E has equation $\frac{x^2}{(a^2 + b^2)} + \frac{y^2}{b^2} = 1$. See diagram below.



i) Show that ellipse E has eccentricity $\frac{1}{e}$. [1]

ii) If H and E intersect at P in the first quadrant, show that the acute angle α between the tangents to H and E at P is given by $\tan\alpha = \sqrt{2}\left(e + \frac{1}{e}\right)$. [6]