Name: Joanna KWOK

Teacher: JOHNSON REPRESENT

Class: ____________________

FORT STREET HIGH SCHOOL

2009

HIGHER SCHOOL CERTIFICATE COURSE

ASSESSMENT TASK 4: TRIAL HSC

Mathematics Extension 2

TIME ALLOWED: 3 HOURS

(PLUS 5 MINUTES READING TIME)

Outcomes Assessed	Questions	Marks
Determines the important features of graphs of a wide variety of functions, including conic sections	4, 5	
Applies appropriate algebraic techniques to complex numbers and polynomials	2, 3	
Applies further techniques of integration, such as slicing and cylindrical shells, integration by parts and recurrence formulae, to problems	1, 6	
Synthesises mathematical solutions to harder problems and communicates them in an appropriate form	7, 8	

Question	1	2	3	4	5	6	7	8	Total	%
Marks	/15 11	/15 10	/15 9	/15 9	/15 11	/15 15	/15 10	/15 4	/120 79	

Directions to candidates:

- Attempt all questions
- The marks allocated for each question are indicated
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board – approved calculators may be used
- Each new question is to be started on a new page

Question 1. Start a NEW booklet (15 marks)

a) Show that $\int_2^4 \frac{dx}{x\sqrt{x-1}} = \frac{\pi}{6}$ 3

b) Find the following indefinite integrals

i) $\int x^3 e^{-x}\, dx$ 3

ii) $\int \frac{x^2-2x+6}{(x^2+4)(x-1)}\, dx$ 3

iii) $\int \frac{2x+5}{\sqrt{x^2+4x+2}}\, dx$ 3

iv) $\int \frac{dx}{2+\sin x}$ 3

Question 2. Start a NEW booklet (15 marks)

a) The complex number ω is given by $-1+i\sqrt{3}$.

i) Show $\omega^2 = 2\overline{\omega}$ 1

ii) Evaluate $|\omega|$ and $\arg \omega$ 2

iii) Show that ω is a root of $\omega^3 - 8 = 0$ 2

b) Sketch the region on the Argand diagram whose points z satisfy the inequalities $|z-\overline{z}| \le 4$ and $\frac{-\pi}{3} \le \arg z \le \frac{\pi}{3}$ 4

c)

i) Prove that $(1+i\tan\theta)^n + (1-i\tan\theta)^n = 2\sec^n(\theta)\cos(n\theta)$ 3

ii) Hence prove $\text{Re}\left(\left(1+i\tan\frac{\pi}{8}\right)^8\right) = 64\left(12\sqrt{2}-17\right)$ 3

Question 3. Start a NEW booklet (15 marks)

a) P and Q are points on the curve $y = x^4 + 4x^3$ where $x = \alpha$ and $x = \beta$ respectively. The line $y = mx + b$ is a tangent to the curve at both points P and Q.

i) By forming an expression for m when $x = \alpha$ and $x = \beta$, show that α and β are double roots of $x^4 + 4x^3 - mx - b = 0$. **2**

ii) Use the relationships between the roots and the coefficients of this equation to find the values of m and b. **3**

b) When a polynomial $P(x)$ is divided by $(x-3)$ the remainder is 5, and when it is divided by $(x-4)$ the remainder is 9.

Find the remainder when $P(x)$ is divided by $(x-4)(x-3)$. **3**

c) The equation $x^3 + kx + 2 = 0$ has roots α, β, γ.

i) Find an expression for $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ in terms of k. **2**

ii) Show that the value of $\alpha^3 + \beta^3 + \gamma^3$ is independent of k. **2**

iii) Find the monic equation with roots $\alpha^2 + \beta^2 + \gamma^2$ and coefficients in terms of k. **3**

Question 4. Start a NEW booklet (15 marks)

a) Consider the curves $\dfrac{x^2}{16}+\dfrac{y^2}{7}=1$ and $x^2-\dfrac{y^2}{8}=1$

i) Show that both curves have the same foci. **2**

ii) Find the equation of the circle through the points of intersection of the two curves. **4**

b) The point $P(x_o, y_0)$ lies on the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$, where $a>b>0$.

i) Write down the equations of the two asymptotes of the hyperbola. **1**

ii) Show that the acute angle α between the two asymptotes satisfies $\tan\alpha=\dfrac{2ab}{a^2-b^2}$ **2**

iii) If M and N are the feet of the perpendiculars drawn from P to the asymptotes, show that $MP.NP=\dfrac{a^2b^2}{a^2+b^2}$ **3**

iv) Hence show that the area of ΔPMN is $\dfrac{a^3b^3}{\left(a^2+b^2\right)^2}$ square units. **3**

Question 5. Start a NEW booklet (15 marks)

a) Consider the function $f(x) = 2 - \frac{4}{x^2+1}$.

i) Show that the function is even. 1

ii) Find the coordinates of any points of intersection with the axes and the equations of any asymptotes of the graph $y = f(x)$. 2

iii) Find the coordinates and nature of any stationary points of $y = f(x)$. 2

iv) Sketch the graph of $y = f(x)$ showing all the above features. 2

v) Draw separate one-third page sketches of the graphs of the following:

α. $y = |f(x)|$ 1

β. $y = [f(x)]^2$ 2

χ. $y^2 = f(x)$ 2

b) If y is a function of x which satisfies the relation $xy = ke^{\frac{y}{x}}$ where k is a constant, show that

$x(x-y)\frac{dy}{dx} + y(x+y) = 0$ 3

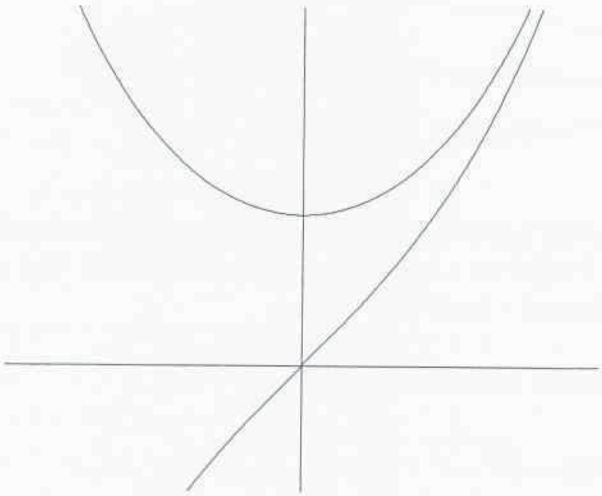
Question 6. Start a NEW booklet (15 marks)

a) The region between the curve $y = 4x - x^2$ and the x-axis is rotated about the line $x = 5$. Find the volume of the solid generated. 4

b) The region bounded by the curves $g(x) = x^3$ and $f(x) = x^4$ is rotated about the x axis.

Using the method of slicing calculate the volume of the solid generated. 5

c) The region between the curves $y = \frac{e^x + e^{-x}}{2}$ and $y = \frac{e^x - e^{-x}}{2}$, the y axis and the line $x = 1$ is rotated about the y axis.



i) Use the method of cylindrical shells to show that the volume of the solid generated is given by $V = 2\pi \int_0^1 xe^{-x} \, dx$ 4

ii) Find the exact value of this volume. 2

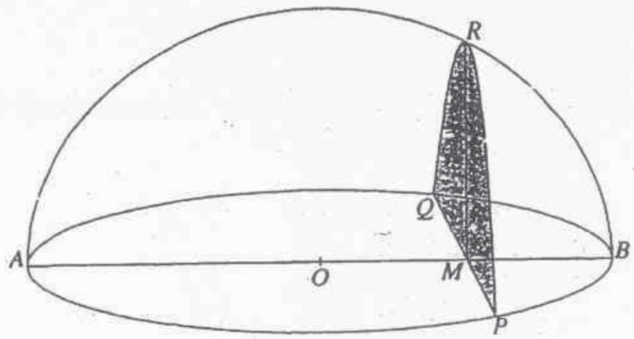
Question 7. Start a NEW booklet (15 marks)

a) A parabola passes through the three points $O(0,0)$, $A(a,h)$ and $B(-a,h)$, where a and h are positive real numbers.

i) Sketch the curve and find its equation. 1

ii) Show that the area contained between the parabola and the line AB is two thirds of the area of the rectangle with vertices *A, B,* $M(a,0)$ and $N(-a,0)$. 3

b)



In the diagram above, a tent has a circular base with centre *O* and radius *l*, and *AOB* is a of the base. The shaded area *PMQR* is a typical cross section of the tent perpendicular to *AB,* and meets *AB* at a point *M* distant *x* from *O.*

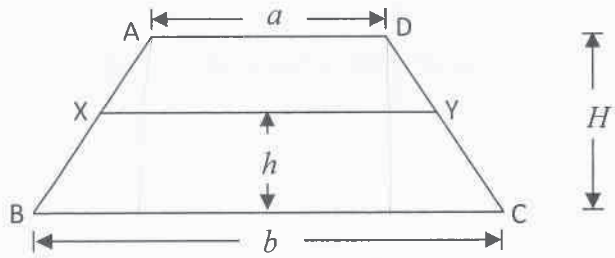
The curve *PRQ* is a parabola with axis *RM* and *QM* = *RM.*

i) Show that $MQ = \sqrt{l^2 - x^2}$ 1

ii) Use part (a) to show that the shaded area *PMQR* is $\frac{4}{3}\left(l^2 - x^2\right)$ 1

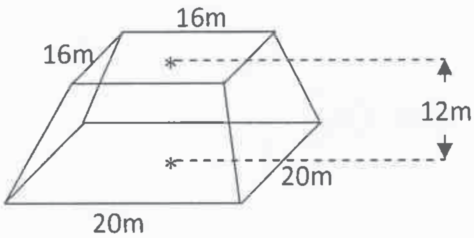
iii) Find the volume of the tent. 3

c) ABCD is an isosceles trapezium of height H with $AB = DC$. The parallel sides AD and BC are of lengths a and b respectively. X and Y are points on AB and DC respectively such that XY is parallel to BC. The perpendicular distance between XY and BC is h.



i) Show that the length of XY is given by $XY = b - \dfrac{(b-a)h}{H}$ **3**

ii) The solid shown has a square base of 20m by 20m and a square top of 16m by 16m.



The top and base lie on two parallel planes. The four sides are isosceles trapeziums. The height of the solid is 12m. Find the volume of the solid by taking slices parallel to the base. **3**

Question 8. Start a NEW booklet (15 marks)

a)

i) Use the substitution $u=\frac{\pi}{4}-x$ to show that $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx=\int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)dx$ 2

ii) Hence find the exact value of $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx$ 1

b)

i) Find $\int \sin(7x)\sin(3x)\,dx$ 3

ii) Find $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ 3

c) If $I_n=\int_0^1 \frac{1}{(x^2+1)^n}\,dx$

i) Prove $I_{n+1}=\frac{1}{2n}\left[2^{-n}+(2n-1)I_n\right]$ 4

ii) Hence evaluate I_3 2