Name: Jason Chiem

Teacher: Mr Hayes

Class: ____________

FORT STREET HIGH SCHOOL

2008

HIGHER SCHOOL CERTIFICATE COURSE

ASSESSMENT TASK 4: TRIAL HSC

Mathematics Extension 2

TIME ALLOWED: 3 HOURS
(PLUS 5 MINUTES READING TIME)

Outcomes Assessed	Questions	Marks
Determines the important features of graphs of a wide variety of functions, including conic sections	4, 5	
Applies appropriate algebraic techniques to complex numbers and polynomials	1, 6	
Applies further techniques of integration, such as slicing and cylindrical shells, integration by parts and recurrence formulae, to problems	2, 7	
Synthesises mathematical solutions to harder problems and communicates them in an appropriate form	3, 8	

Question	1	2	3	4	5	6	7	8	Total	%
Marks	14/15	13/15	14/15	15/15	13/15	12/15	15/15	11/15	/120	

Directions to candidates:

- Attempt all questions
- The marks allocated for each question are indicated
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board – approved calculators may be used
- Each new question is to be started on a new page

Question 1 (15 Marks) **Marks**

a) Find the modulus and argument of the complex numbers w and z **4**

Where $z = \dfrac{1+i}{1-i}$ and $w = \dfrac{\sqrt{2}}{1-i}$

b) Plot the points $z,\ w,\ z+w$ from part a) on an accurate **3**

Argand diagram and hence find the exact value of $\tan(\dfrac{3\pi}{8})$.

c) The vertices of a square taken anticlockwise are P, Q, R and S. **2**
If the points P and Q are represented by the complex numbers
$z_P = -1+4i$ and $z_Q = -3$
Find the other corners of the square R and S and its centre
in the form a+ib .

d) Determine the greatest and least values of arg z, **2**
when $|z-8i-5| = 6$, answer to the nearest minute.

e) In the Argand plane **4**

(i) shade: $|z+3|+|z-3| \le 10$ *and* $3 \le |z-3+2i| \le 4$

(ii) sketch: $\arg(z-5)-\arg(z+3) = \dfrac{\pi}{4}$

Question 2 (15 Marks) **Marks**

a) Show $\int_1^2 \frac{1}{x^2} \ln(x+1)\, dx = \frac{1}{2}\ln\frac{4}{3} + \int_1^2 \frac{1}{x(x+1)}\, dx$ **4**

and hence evaluate

$\int_1^2 \frac{1}{x^2} \ln(x+1)\, dx$ leaving answer in simplest exact form.

b) Simplify $\frac{1}{1-\sin x} - \frac{1}{1+\sin x}$ and **3**

hence find

$$\int \frac{1}{1-\sin x} - \frac{1}{1+\sin x}\, dx$$

c) Find $\int \frac{1}{\sqrt{16-25x^2}}\, dx$ **2**

d) Evaluate $\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\, dx$ leave answer in exact form. **2**

e) Find $\int \frac{-2x+4}{(x^2+1)(x-1)^2}\, dx$ **4**

Question 3 (15 Marks) **Marks**

a) Let $t = \tan\frac{\theta}{2}$

(i) Find expressions for $\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$ in terms of t **2**

(ii) Hence show $\sin\theta = \frac{2t}{1+t^2}$ and $\cos\theta = \frac{1-t^2}{1+t^2}$ **2**

(iii) Show that $\frac{dt}{d\theta} = \frac{1}{2}(1+t^2)$ **1**

(iv) Hence evaluate $\int_0^{\frac{\pi}{2}} \frac{d\theta}{5+3\sin\theta+4\cos\theta}$ **4**

b) If $I_n = \int x^n(2x+c)^{-\frac{1}{2}}\,dx$, show that

(i) $I_n = \frac{x^n(2x+c)^{\frac{1}{2}}}{2n+1} - \frac{ncI_{n-1}}{2n+1}$ **3**

(ii) Hence evaluate $\int_0^1 x^3(2x+1)^{-\frac{1}{2}}\,dx$ **3**

Question 4 (15 Marks)

Marks

a) The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with a>b and eccentricity e.

The foci of the ellipse are S and S' and M , M' are the feet of the perpendiculars from P onto the directrices corresponding to S and S'.

The Normal to the ellipse at P meets the major axis of the ellipse At H.

(i) Draw a sketch to illustrate the above information. **2**

(ii) Prove $SP+S'P=2a$. **1**

(iii) Show that the coordinates of H are $\left(\frac{(a^2-b^2)\cos\theta}{a}\ ,\ 0\right)$. **3**

(iv) Show that $\frac{HS}{HS'}=\frac{1-e\cos\theta}{1+e\cos\theta}=\frac{PS}{PS'}$ **2**

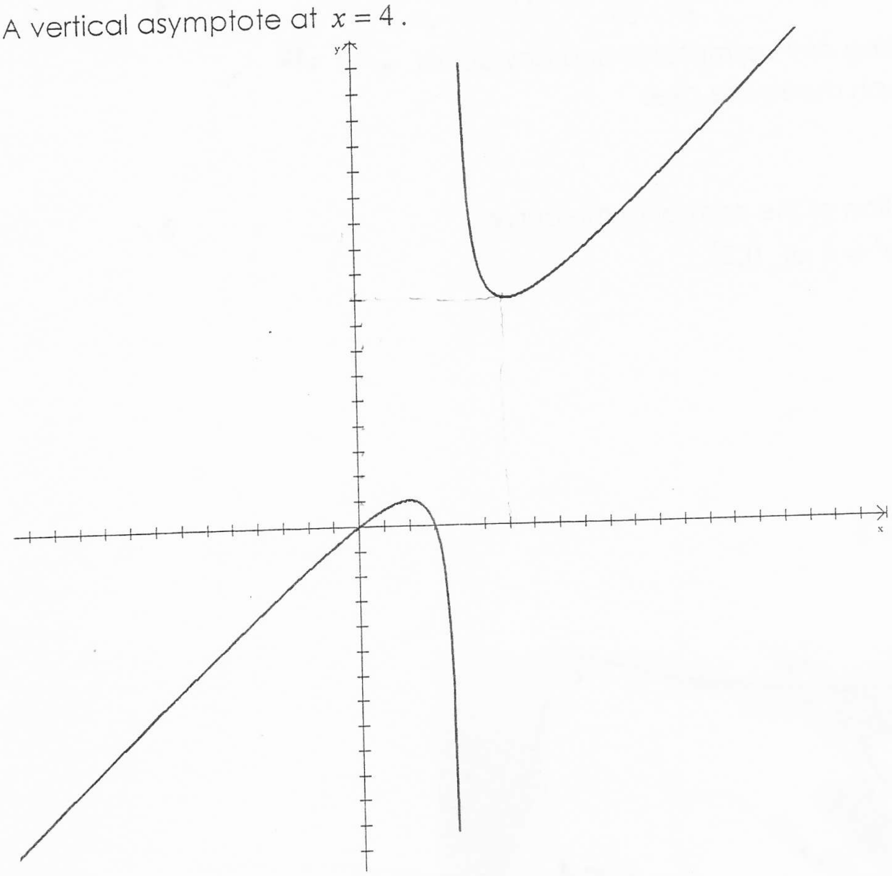
b) Show that the locus of the point $Q\left\{\frac{a}{2}(t+\frac{1}{t}),\frac{b}{2}(t-\frac{1}{t})\right\}$ for varying values of t is the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. **2**

(I) Show the gradient of the tangent at Q is $\frac{b}{a}\left(\frac{t^2+1}{t^2-1}\right)$ **2**

(II) Derive the equation of the tangent at Q **3**

Question 5 (15 Marks) **Marks**

a) The diagram shows the graph of $y = f(x)$. The graph has A vertical asymptote at $x = 4$.



Draw separate one third page sketches of the graphs of the following

(i) $y = \sqrt{f(x)}$ **2**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y^2 = f(x)$ **2**

(iv) $y = \cos(f(x))$ **2**

This question continues on the next page

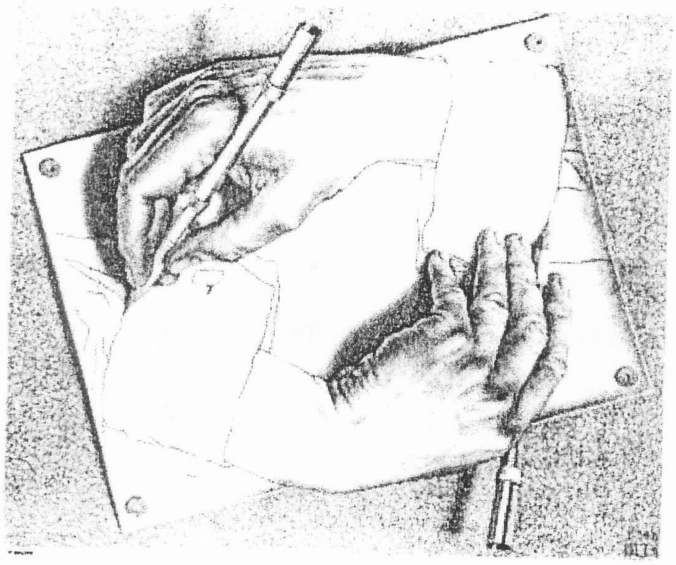
Question 5 continued

Marks

b) Sketch the graph of $y = x + \dfrac{x}{x^2 - 25}$ clearly indicating any asymptotes and any points where the graph meets the axes — 4

c) Find the equation of the normal to the curve $x^3y - 3xy^2 + 2y^3 = 6$ at $(1,2)$ — 3

End of Question 5



Question 6 (15 Marks) **Marks**

a) If $\frac{p}{q}$ is a zero of the polynomial (p and q are relatively prime)

$P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 \ , a_n \neq 0$

and $a_0, a_1, a_2, a_3, ...a_n$ are integers,

(i) Show q/a_n (q divides a_n) and p/a_0 (p divides a_0) **2**

(ii) Given $P(x) = x^3 - 4x^2 - 3x - 10$ ha s a rational root, factor $P(x)$ over the complex field. **3**

b) Show that if the polynomial $P(x)$ has a root of α multiplicity m, then $P'(x)$ has a root of multiplicity $m-1$. **1**

Given that $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$ has a three fold root, Find all the roots of $P(x)$. **2**

c) If α, β, δ are the roots of $p(x) = 2x^3 - 4x^2 - 3x - 1$ Find the values of $\alpha^3 + \beta^3 + \delta^3$. **2**

d) Let $f(t) = t^3 + ct + d$ where c and d are constants
Suppose that the equation $f(t) = 0$ has three distinct real roots $t_1, t_2, \text{and } t_3$.

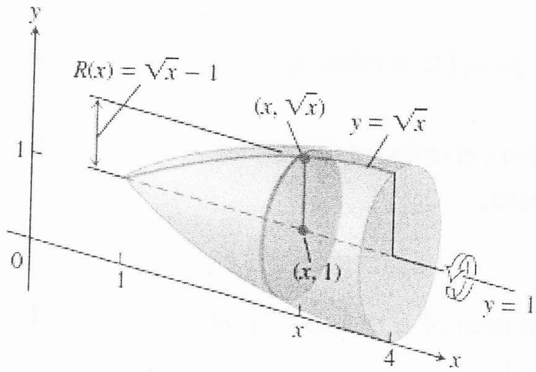
(i) Show that $t_1^2 + t_2^2 + t_3^2 = -2c$ **2**

(ii) If the function $y = f(t)$ has two turning points at $t = u$ and $t = \text{v}$ and $f(u) \times f(v) < 0$
Show that $27d^2 + 4c^3 < 0$. **3**

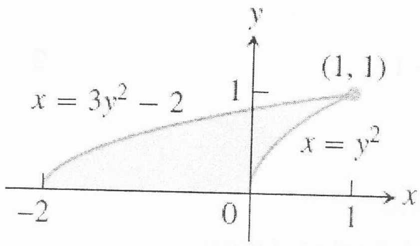
Question 7 (15 Marks)

Marks

a) Find the volume of the solid generated by revolving the region bounded By $y=\sqrt{x}$ and the lines $x=1,\ x=4$ about the line $y=1$. Use the slicing method. **2**



b) The region shown here is revolved about the x-axis to generate a solid. **3**



Use the method of cylindrical shells to find the volume

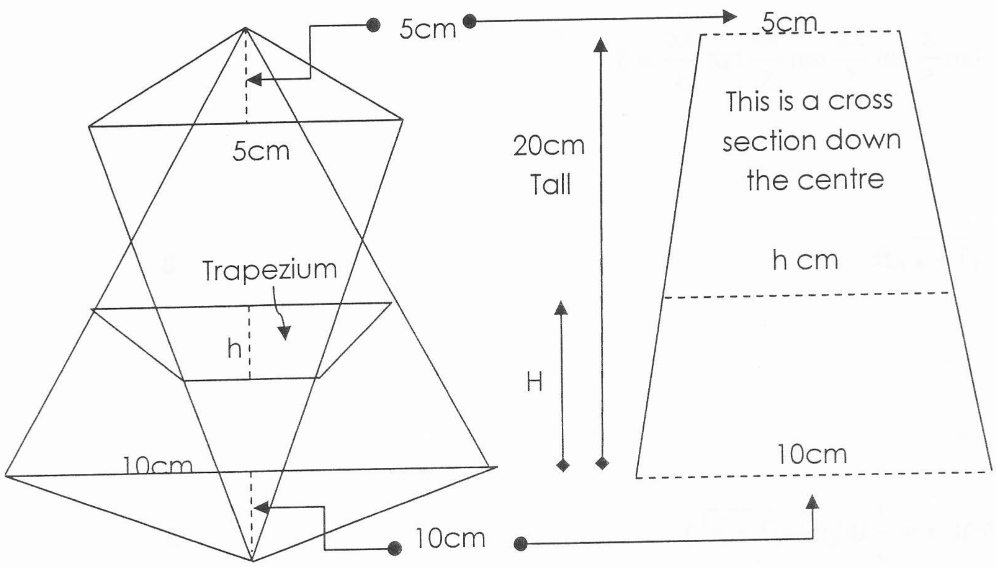
c) The circle $(x-6)^2+(y-4)^2=4$ is rotated around the line $x=2$. Calculate the exact volume generated. **5**

This question is continued on the next page

Question 7 continued

Marks

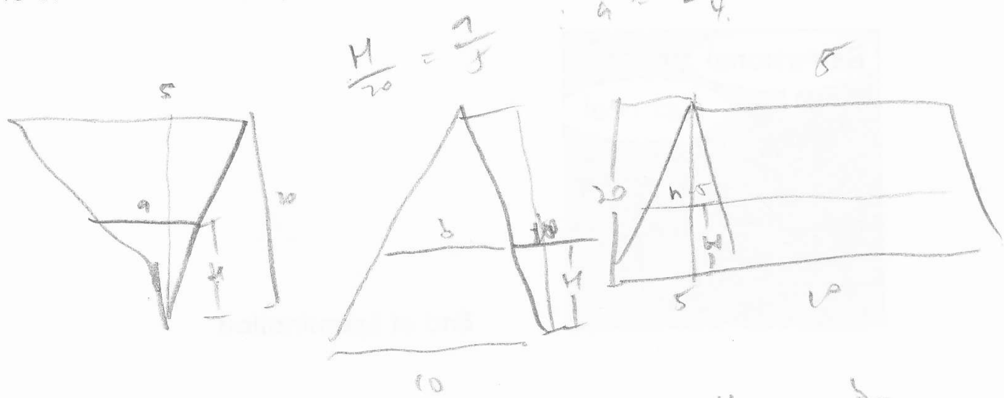
d) A Saltshaker 20 cm tall is made with isosceles triangular ends and a cross section which is an isosceles trapezium. **5**
Note top and bottom triangles have bases and perpendicular heights equal.



The Trapezium is located H cm above the base, show using similarity

that the trapezium has an area of $A = 50 - \frac{10H}{4} + \frac{H^2}{32}$ cm^2

Hence find the volume of the saltshaker to the nearest millilitre.



$\frac{20-H}{20} = \frac{b}{10}$

$b = \frac{20-H}{2}$

Question 8 (15 Marks) **Marks**

a) Use De Moivre's theorem to express $\cos 5\theta,\ \sin 5\theta$ in terms of $\sin\theta$ and $\cos\theta$. **5**

Hence express $\tan 5\theta$ as a rational function of t where $t = \tan\theta$.

Deduce that $\tan\frac{\pi}{5}\tan\frac{2\pi}{5}\tan\frac{3\pi}{5}\tan\frac{4\pi}{5} = 5.$

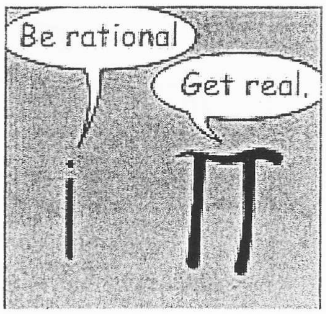
b) Find $\int \ln(\sqrt{x}+\sqrt{1+x})dx$ **3**

c) If $y = \frac{1}{2}(e^{ax} - e^{-ax})$

(i) Show that $x = \frac{1}{a}\ln(y+\sqrt{1+y^2})$ **2**

(ii) Show that $\left(\frac{dy}{dx}\right)^2 - a^2y^2 = a^2$ **2**

(iii) Hence deduce that $\int \frac{dy}{\sqrt{1+y^2}} = \log_e(y+\sqrt{1+y^2})+c$ **3**



End of Examination