A formula for e from the product of binomial coefficients

by Derek Buchanan

Using a calculator we can check

$\binom{1}{0}^2 \cdot \binom{1}{1}^2 = 1$

$\sqrt{\sqrt{\binom{2}{0}^2} \cdot \sqrt{\binom{2}{1}^2} \cdot \sqrt{\binom{2}{2}^2}} = 1.414213562\ldots$

$\sqrt[3]{\sqrt[3]{\binom{3}{0}^2} \cdot \sqrt[3]{\binom{3}{1}^2} \cdot \sqrt[3]{\binom{3}{2}^2} \cdot \sqrt[3]{\binom{3}{3}^2}} = 1.629498222\ldots$ etc.,

or writing the last one in Pi notation, $\sqrt[3]{\prod_{r=0}^{3} \sqrt[3]{\binom{3}{r}^2}} = 1.629498222\ldots$

and then with wolframalpha we can check:

$\sqrt[10]{\prod_{r=0}^{10} \sqrt[10]{\binom{10}{r}^2}} = 2.139008611\ldots$

$\sqrt[100]{\prod_{r=0}^{100} \sqrt[100]{\binom{100}{r}^2}} = 2.599090034\ldots$

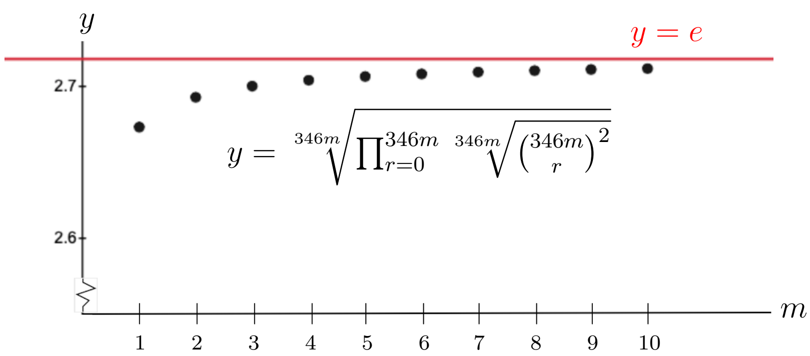
$\sqrt[1000]{\prod_{r=0}^{1000} \sqrt[1000]{\binom{1000}{r}^2}} = 2.699991845\ldots$

and as $n = 3460$ seems to be the limit of wolframalpha we check

$\sqrt[3460]{\prod_{r=0}^{3460} \sqrt[3460]{\binom{3460}{r}^2}} = 2.712012851\ldots$

You may begin to see what is happening here if you know that $e = 2.718281828\ldots$.

The graph below shows it as well for the functions $y = \sqrt[346m]{\prod_{r=0}^{346m} \sqrt[346m]{\binom{346m}{r}^2}}$ for $m = 1, 2, 3, \ldots, 10$ and $y = e$.



Points
$(1, 2.673879)$
$(2, 2.693315)$
$(3, 2.700563)$
$(4, 2.704422)$
$(5, 2.706840)$
$(6, 2.708507)$
$(7, 2.709729)$
$(8, 2.710666)$
$(9, 2.711409)$
$(10, 2.712013)$

We can ascertain a formula for e, $\boxed{\lim_{n\to\infty} \sqrt[n]{\prod_{r=0}^{n} \sqrt[n]{\binom{n}{r}^2}} = e}$ which will follow from the

Proposition. For all real numbers α, β, $\lim_{n\to\infty} \left(\prod_{r=0}^{n} \binom{n}{r}\right)^{\frac{1}{(n+\alpha)(n+\beta)}} = \sqrt{e}$

Proof (integration method).

If $p_n = \ln \prod_{r=0}^{n} \binom{n}{r} = \ln \prod_{r=1}^{n} r^{2r-n-1} = \sum_{r=1}^{n} (2r-n) \ln \frac{r}{n} + n \ln n - \ln n!$

noting that $\lim_{n\to\infty} \frac{n \ln n}{n^2} = \lim_{n\to\infty} \frac{\ln n!}{n^2} = 0$ then

$$\begin{aligned}
\ln \lim_{n\to\infty} \left(\prod_{r=0}^{n} \binom{n}{r}\right)^{\frac{1}{(n+\alpha)(n+\beta)}} &= \lim_{n\to\infty} \tfrac{p_n}{(n+\alpha)(n+\beta)} \\
&= \lim_{n\to\infty} \tfrac{p_n}{n^2} \\
&= \lim_{n\to\infty} \tfrac{1}{n^2} \textstyle\sum_{r=1}^{n} (2r-n) \ln \tfrac{r}{n} \\
&= \lim_{n\to\infty} \textstyle\sum_{r=1}^{n} \left(\tfrac{2r}{n} - 1\right) \left(\ln \tfrac{r}{n}\right) \cdot \tfrac{1}{n} \\
&= \textstyle\int_0^1 (2x-1) \ln x \, dx \\
&= \tfrac{1}{2}
\end{aligned}$$

Hence $\lim_{n\to\infty} \left(\prod_{r=0}^{n} \binom{n}{r}\right)^{\frac{1}{(n+\alpha)(n+\beta)}} = e^{\frac{1}{2}} = \sqrt{e}$ □

Now with $\alpha = \beta = 0$ this can be rearranged to get the formula $\lim_{n\to\infty} \sqrt[n]{\prod_{r=0}^{n} \sqrt[n]{\binom{n}{r}^2}} = e$.

What seems to be more interesting however is the fact that there are many other ways to prove the proposition. I chose the integration method first because it seems to be more closely aligned with the NSW Syllabuses than the others.

Here are some other proofs using l'Hôpital's rule [1], Stirling's formula [2] and the Stolz–Cesàro theorem [3].

Proof (l'Hôpital's rule method).

Where H = the hyperfactorial function, Γ = the Gamma function, ψ = the digamma function and $\psi^{(1)}$ = the trigamma function,

$$\begin{aligned}
\ln \lim_{n\to\infty} \left(\prod_{r=0}^{n} \binom{n}{r}\right)^{\frac{1}{(n+\alpha)(n+\beta)}} &= \ln \left(\lim_{n\to\infty} \prod_{r=1}^{n} r^{2r-n-1}\right)^{\frac{1}{(n+\alpha)(n+\beta)}} \\
&= \lim_{n\to\infty} \textstyle\sum_{r=1}^{n} \tfrac{(2r-n-1)\ln r}{(n+\alpha)(n+\beta)} \\
&= \lim_{x\to\infty} \tfrac{2\ln H(x) - (x+1)\ln\Gamma(x+1)}{x^2+(\alpha+\beta)x+\alpha\beta} \\
&= \lim_{x\to\infty} \tfrac{\frac{d}{dx}(2\ln H(x) - (x+1)\ln\Gamma(x+1))}{\frac{d}{dx}(x^2+(\alpha+\beta)x+\alpha\beta)} \text{ using l'Hôpital's rule} \\
&= \lim_{x\to\infty} \tfrac{\ln\Gamma(x+1)+2x-(x+1)\psi(x+1)-\ln(2\pi)+1}{2x+\alpha+\beta} \\
&= \lim_{x\to\infty} \tfrac{\frac{d}{dx}(\ln\Gamma(x+1)+2x-(x+1)\psi(x+1)-\ln(2\pi)+1)}{\frac{d}{dx}(2x+\alpha+\beta)} \text{ by l'Hôpital again} \\
&= \lim_{x\to\infty} \left(1 - \tfrac{1}{2}(x+1)\psi^{(1)}(x+1)\right) \\
&= \lim_{x\to\infty} \left(1 - \tfrac{1}{2}\left(1 + \tfrac{1}{2x} - \tfrac{1}{3x^2} + \tfrac{1}{6x^3} - \tfrac{1}{30x^4} + \cdots\right)\right) \\
&\qquad \text{by Laurent series expansion} \\
&= 1 - \tfrac{1}{2} \\
&= \tfrac{1}{2}
\end{aligned}$$

Hence $\lim_{n\to\infty} \left(\prod_{r=0}^{n} \binom{n}{r}\right)^{\frac{1}{(n+\alpha)(n+\beta)}} = e^{\frac{1}{2}} = \sqrt{e}$. □

Proof (Stirling's formula method).

If $p_n = \left(\prod_{r=0}^{n} \binom{n}{r}\right)^{\frac{1}{(n+\alpha)(n+\beta)}}$ then $p_n^{(n+\alpha)(n+\beta)} = p_{n-1}^{(n+\alpha-1)(n+\beta-1)} \cdot \frac{n^n}{n!}$.

If $L = \lim_{n\to\infty} p_n$ then

$$\lim_{n\to\infty} \frac{L^{(n+\alpha)(n+\beta)}}{L^{(n+\alpha-1)(n+\beta-1)}\cdot\frac{n^n}{n!}} = \lim_{n\to\infty} \frac{L^{2n+\alpha+\beta-1}n!}{n^n}$$
$$= \lim_{n\to\infty}\left(L^{2n+\alpha+\beta-1}\cdot\frac{\sqrt{2\pi n}}{e^n}\right)\cdot\lim_{n\to\infty}\left(\frac{n!}{n^n}\cdot\frac{e^n}{\sqrt{2\pi n}}\right)$$
$$= \lim_{n\to\infty}\left(L^{2n+\alpha+\beta-1}\cdot\frac{\sqrt{2\pi n}}{e^n}\right)\cdot 1 \text{ by Stirling's formula}$$
$$= 1$$
$$\therefore L = \lim_{n\to\infty}\left(\frac{e^n}{\sqrt{2\pi n}}\right)^{\frac{1}{2n+\alpha+\beta-1}}$$
$$= \lim_{n\to\infty}\left(\frac{e^n}{\sqrt{2\pi n}}\right)^{\frac{1}{2n}}$$
$$= \sqrt{e}\cdot\lim_{n\to\infty}\left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{1}{2n}}\cdot\sqrt[4]{\lim_{x\to 0^+} x^x} \text{ if } x=\frac{1}{n}$$
$$= \sqrt{e}\cdot 1\cdot 1$$
$$= \sqrt{e} \qquad \square$$

Proof (Stolz–Cesàro theorem method).

$$\ln\lim_{n\to\infty}\left(\prod_{r=0}^{n}\binom{n}{r}\right)^{\frac{1}{(n+\alpha)(n+\beta)}} = \ln\left(\lim_{n\to\infty}\prod_{r=1}^{n} r^{2r-n-1}\right)^{\frac{1}{(n+\alpha)(n+\beta)}}$$
$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{(2r-n-1)\ln r}{(n+\alpha)(n+\beta)}$$
$$= \lim_{n\to\infty}\frac{2\ln\prod_{r=1}^{n} r^r-(n+1)\ln n!}{(n+\alpha)(n+\beta)}$$
$$= \lim_{n\to\infty}\frac{2\ln\prod_{r=1}^{n+1} r^r-(n+2)\ln(n+1)!-(2\ln\prod_{r=1}^{n} r^r-(n+1)\ln n!)}{(n+\alpha+1)(n+\beta+1)-(n+\alpha)(n+\beta)}$$

using the Stolz-Cesàro theorem

$$= \lim_{n\to\infty}\frac{\ln(n+1)^{n+1}-\ln(n+1)!}{2(n+1)+\alpha+\beta-1}$$
$$= \lim_{n\to\infty}\frac{\ln(n+1)^{n+1}-\ln(n+1)!-(\ln n^n-\ln n!)}{2(n+1)+\alpha+\beta-1-(2n+\alpha+\beta-1)}$$

by Stolz-Cesàro again

$$= \frac{1}{2}\lim_{n\to\infty}\ln(1+\tfrac{1}{n})^n$$
$$= \frac{1}{2}\ln e$$
$$= \frac{1}{2}$$

Hence $\lim_{n\to\infty}\left(\prod_{r=0}^{n}\binom{n}{r}\right)^{\frac{1}{(n+\alpha)(n+\beta)}} = e^{\frac{1}{2}} = \sqrt{e}$ $\square$

I'm sure there are many other ways to prove it, but these are a good start.

References.

[1] https://en.wikipedia.org/wiki/L'Hôpital's_rule

[2] https://en.wikipedia.org/wiki/Stirling's_approximation

[3] https://en.wikipedia.org/wiki/Stolz–Cesàro_theorem