

A formula for e from the product of binomial coefficients

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Using a calculator we can check

$$\binom{1}{0}^2 \cdot \binom{1}{1}^2 = 1$$

$$\sqrt{\sqrt{\binom{2}{0}^2} \cdot \sqrt{\binom{2}{1}^2} \cdot \sqrt{\binom{2}{2}^2}} = 1.414213562\dots$$

$$\sqrt[3]{\sqrt[3]{\binom{3}{0}^2} \cdot \sqrt[3]{\binom{3}{1}^2} \cdot \sqrt[3]{\binom{3}{2}^2} \cdot \sqrt[3]{\binom{3}{3}^2}} = 1.629498222\dots \text{ etc.,}$$

$$\text{or writing the last one in Pi notation, } \sqrt[3]{\prod_{r=0}^3 \sqrt[3]{\binom{3}{r}^2}} = 1.629498222\dots$$

and then with wolframalpha we can check:

$$\sqrt[10]{\prod_{r=0}^{10} \sqrt[10]{\binom{10}{r}^2}} = 2.139008611\dots$$

$$\sqrt[100]{\prod_{r=0}^{100} \sqrt[100]{\binom{100}{r}^2}} = 2.599090034\dots$$

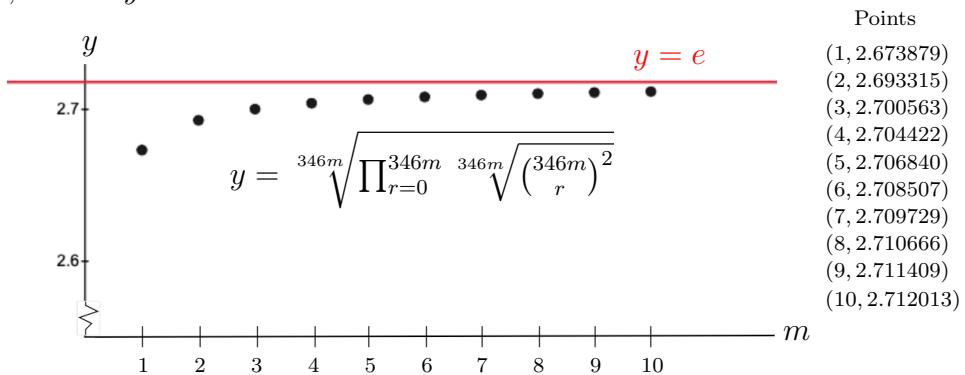
$$\sqrt[1000]{\prod_{r=0}^{1000} \sqrt[1000]{\binom{1000}{r}^2}} = 2.699991845\dots$$

and as $n = 3460$ seems to be the limit of wolframalpha we check

$$\sqrt[3460]{\prod_{r=0}^{3460} \sqrt[3460]{\binom{3460}{r}^2}} = 2.712012851\dots$$

You may begin to see what is happening here if you know that $e = 2.718281828\dots$

The graph below shows it as well for the functions $y = \sqrt[346m]{\prod_{r=0}^{346m} \sqrt[346m]{\binom{346m}{r}^2}}$ for $m = 1, 2, 3, \dots, 10$ and $y = e$.



We can ascertain a formula for e , $\lim_{n \rightarrow \infty} \sqrt[n]{\prod_{r=0}^n \sqrt[n]{\binom{n}{r}^2}} = e$ which will follow from the

Proposition. For all real numbers α, β , $\lim_{n \rightarrow \infty} \left(\prod_{r=0}^n \binom{n}{r} \right)^{\frac{1}{(n+\alpha)(n+\beta)}} = \sqrt{e}$

Proof (integration method).

If $p_n = \ln \prod_{r=0}^n \binom{n}{r} = \ln \prod_{r=1}^n r^{2r-n-1} = \sum_{r=1}^n (2r-n) \ln \frac{r}{n} + n \ln n - \ln n!$

noting that $\lim_{n \rightarrow \infty} \frac{n \ln n}{n^2} = \lim_{n \rightarrow \infty} \frac{\ln n!}{n^2} = 0$ then

$$\begin{aligned}
\ln \lim_{n \rightarrow \infty} \left(\prod_{r=0}^n \binom{n}{r} \right)^{\frac{1}{(n+\alpha)(n+\beta)}} &= \lim_{n \rightarrow \infty} \frac{p_n}{(n+\alpha)(n+\beta)} \\
&= \lim_{n \rightarrow \infty} \frac{p_n}{n^2} \\
&= \lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{r=1}^n (2r-n) \ln \frac{r}{n} \\
&= \lim_{n \rightarrow \infty} \sum_{r=1}^n \left(\frac{2r}{n} - 1 \right) \left(\ln \frac{r}{n} \right) \cdot \frac{1}{n} \\
&= \int_0^1 (2x-1) \ln x \, dx \\
&= \frac{1}{2}
\end{aligned}$$

Hence $\lim_{n \rightarrow \infty} \left(\prod_{r=0}^n \binom{n}{r} \right)^{\frac{1}{(n+\alpha)(n+\beta)}} = e^{\frac{1}{2}} = \sqrt{e}$ \square

Now with $\alpha = \beta = 0$ this can be rearranged to get the formula $\lim_{n \rightarrow \infty} \sqrt[n]{\prod_{r=0}^n \sqrt[n]{\binom{n}{r}^2}} = e$.

What seems to be more interesting however is the fact that there are many other ways to prove the proposition. I chose the integration method first because it seems to be more closely aligned with the NSW Syllabuses than the others.

Here are some other proofs using l'Hôpital's rule [1], Stirling's formula [2] and the Stolz–Cesàro theorem [3].

Proof (l'Hôpital's rule method).

Where H = the hyperfactorial function, Γ = the Gamma function, ψ = the digamma function and $\psi^{(1)}$ = the trigamma function,

$$\begin{aligned}
\ln \lim_{n \rightarrow \infty} \left(\prod_{r=0}^n \binom{n}{r} \right)^{\frac{1}{(n+\alpha)(n+\beta)}} &= \ln \left(\lim_{n \rightarrow \infty} \prod_{r=1}^n r^{2r-n-1} \right)^{\frac{1}{(n+\alpha)(n+\beta)}} \\
&= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{(2r-n-1) \ln r}{(n+\alpha)(n+\beta)} \\
&= \lim_{x \rightarrow \infty} \frac{2 \ln H(x) - (x+1) \ln \Gamma(x+1)}{x^2 + (\alpha+\beta)x + \alpha\beta} \\
&= \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}(2 \ln H(x) - (x+1) \ln \Gamma(x+1))}{\frac{d}{dx}(x^2 + (\alpha+\beta)x + \alpha\beta)} \text{ using l'Hôpital's rule} \\
&= \lim_{x \rightarrow \infty} \frac{\ln \Gamma(x+1) + 2x - (x+1)\psi(x+1) - \ln(2\pi) + 1}{2x + \alpha + \beta} \\
&= \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}(\ln \Gamma(x+1) + 2x - (x+1)\psi(x+1) - \ln(2\pi) + 1)}{\frac{d}{dx}(2x + \alpha + \beta)} \text{ by l'Hôpital again} \\
&= \lim_{x \rightarrow \infty} (1 - \frac{1}{2}(x+1)\psi^{(1)}(x+1)) \\
&= \lim_{x \rightarrow \infty} (1 - \frac{1}{2}(1 + \frac{1}{2x} - \frac{1}{3x^2} + \frac{1}{6x^3} - \frac{1}{30x^4} + \dots)) \\
&\quad \text{by Laurent series expansion} \\
&= 1 - \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

Hence $\lim_{n \rightarrow \infty} \left(\prod_{r=0}^n \binom{n}{r} \right)^{\frac{1}{(n+\alpha)(n+\beta)}} = e^{\frac{1}{2}} = \sqrt{e}$. \square

Proof (Stirling's formula method).

If $p_n = \left(\prod_{r=0}^n \binom{n}{r} \right)^{\frac{1}{(n+\alpha)(n+\beta)}}$ then $p_n^{(n+\alpha)(n+\beta)} = p_{n-1}^{(n+\alpha-1)(n+\beta-1)} \cdot \frac{n^n}{n!}$.

If $L = \lim_{n \rightarrow \infty} p_n$ then

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{L^{(n+\alpha)(n+\beta)}}{L^{(n+\alpha-1)(n+\beta-1)} \cdot \frac{n^n}{n!}} &= \lim_{n \rightarrow \infty} \frac{L^{2n+\alpha+\beta-1} n!}{n^n} \\
&= \lim_{n \rightarrow \infty} \left(L^{2n+\alpha+\beta-1} \cdot \frac{\sqrt{2\pi n}}{e^n} \right) \cdot \lim_{n \rightarrow \infty} \left(\frac{n!}{n^n} \cdot \frac{e^n}{\sqrt{2\pi n}} \right) \\
&= \lim_{n \rightarrow \infty} \left(L^{2n+\alpha+\beta-1} \cdot \frac{\sqrt{2\pi n}}{e^n} \right) \cdot 1 \text{ by Stirling's formula} \\
&= 1 \\
\therefore L &= \lim_{n \rightarrow \infty} \left(\frac{e^n}{\sqrt{2\pi n}} \right)^{\frac{1}{2n+\alpha+\beta-1}} \\
&= \lim_{n \rightarrow \infty} \left(\frac{e^n}{\sqrt{2\pi n}} \right)^{\frac{1}{2n}} \\
&= \sqrt{e} \cdot \lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{2\pi}} \right)^{\frac{1}{2n}} \cdot \sqrt[4]{\lim_{x \rightarrow 0^+} x^x} \text{ if } x = \frac{1}{n} \\
&= \sqrt{e} \cdot 1 \cdot 1 \\
&= \sqrt{e} \quad \square
\end{aligned}$$

Proof (Stolz–Cesàro theorem method).

$$\begin{aligned}
\ln \lim_{n \rightarrow \infty} \left(\prod_{r=0}^n \binom{n}{r} \right)^{\frac{1}{(n+\alpha)(n+\beta)}} &= \ln \left(\lim_{n \rightarrow \infty} \prod_{r=1}^n r^{2r-n-1} \right)^{\frac{1}{(n+\alpha)(n+\beta)}} \\
&= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{(2r-n-1) \ln r}{(n+\alpha)(n+\beta)} \\
&= \lim_{n \rightarrow \infty} \frac{2 \ln \prod_{r=1}^n r^r - (n+1) \ln n!}{(n+\alpha)(n+\beta)} \\
&= \lim_{n \rightarrow \infty} \frac{2 \ln \prod_{r=1}^{n+1} r^r - (n+2) \ln(n+1)! - (2 \ln \prod_{r=1}^n r^r - (n+1) \ln n!)}{(n+\alpha+1)(n+\beta+1) - (n+\alpha)(n+\beta)} \\
&\quad \text{using the Stolz-Cesàro theorem} \\
&= \lim_{n \rightarrow \infty} \frac{\ln(n+1)^{n+1} - \ln(n+1)!}{2(n+1)+\alpha+\beta-1} \\
&= \lim_{n \rightarrow \infty} \frac{\ln(n+1)^{n+1} - \ln(n+1)! - (\ln n^n - \ln n!)}{2(n+1)+\alpha+\beta-1 - (2n+\alpha+\beta-1)} \\
&\quad \text{by Stolz-Cesàro again} \\
&= \frac{1}{2} \lim_{n \rightarrow \infty} \ln(1 + \frac{1}{n})^n \\
&= \frac{1}{2} \ln e \\
&= \frac{1}{2}
\end{aligned}$$

$$\text{Hence } \lim_{n \rightarrow \infty} \left(\prod_{r=0}^n \binom{n}{r} \right)^{\frac{1}{(n+\alpha)(n+\beta)}} = e^{\frac{1}{2}} = \sqrt{e} \quad \square$$

I'm sure there are many other ways to prove it, but these are a good start.

References.

- [1] https://en.wikipedia.org/wiki/L'Hôpital's_rule
- [2] https://en.wikipedia.org/wiki/Stirling's_approximation
- [3] https://en.wikipedia.org/wiki/Stolz–Cesàro_theorem