QUESTION 1. [15 Marks]

(a) (i) Find $\int \frac{\sqrt{1-x^2}}{\sqrt{1-x}}\,dx$ **2**

(ii) Find $\int x\cos x\,dx$ **2**

(b) Evaluate in simplest exact form $\int_0^4 \frac{8-2x}{(1+x)(4+x^2)}\,dx$ **3**

(c) Use the substitution $t=\tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{2}{5+3\cos x}\,dx$, giving the answer correct to 2 significant figures. **3**

(d) (i) Use the substitution $u=\frac{\pi}{4}-x$ to show that **3**

$$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)dx$$

(ii) Hence find the exact value of $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx$ **2**

QUESTION 2. [15 Marks] [START A NEW PAGE]

(a) Solve the equation $z^2 + 2\overline{z} + 6 = 0$, giving the solutions in the form $z = a + ib$ where a and b are real. **4**

(b) $z_1 = 2\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$ and $z_2 = 2i$ are two complex numbers.

(i) On an Argand diagram draw the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ to represent z_1 and z_2 respectively. Also draw the vectors $z_1 + z_2$ and $z_2 - z_1$. **2**

(ii) Find the exact values of $\arg(z_1 + z_2)$ and $\arg(z_2 - z_1)$. **2**

(c) (i) Express $z = 2\sqrt{3} - 2i$ in modulus-argument form. **2**

(ii) Hence find the values of $z^{\frac{1}{2}}$ in modulus-argument form. **2**

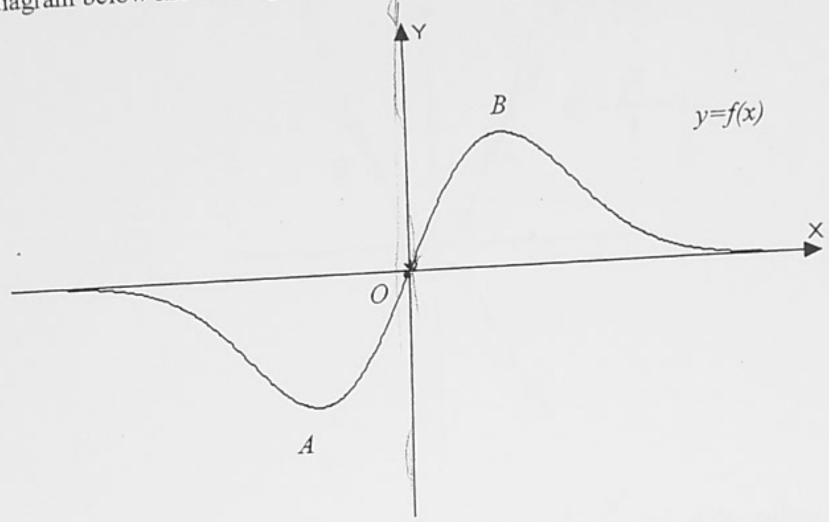
(d) The point P representing the complex number z moves in an Argand diagram so that $|z| = |z - 4 + 2i|$.

(i) Show that the locus of P has equation $2x - y - 5 = 0$. **2**

(ii) Hence find the minimum value of $|z|$. **1**

QUESTION 3. [15 Marks] [START A NEW PAGE]

(a) The diagram below shows the graph of the function $f(x) = xe^{-\frac{1}{2}x^2}$.



(i) Find the coordinates of the stationary points A and B. **2**

(ii) Find the gradient of the tangent to the curve at the origin O. Hence find the set of values of the real number k such that the equation $f(x) = kx$ has three real roots. **2**

(iii) On separate diagrams sketch the following graphs, showing the coordinates of any stationary points:

$(\alpha)\ y = |f(x)|$ **1**

$(\beta)\ y = \{f(x)\}^2$ **2**

$(\gamma)\ y^2 = f(x)$ **2**

(b) (i) Show that $\frac{d}{dx}\{\tan^{-1} e^x + \tan^{-1} e^{-x}\} = 0$. **2**

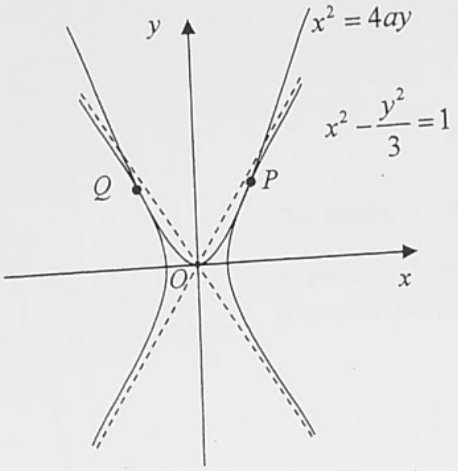
(ii) Hence show that $\tan^{-1} e^x + \tan^{-1} e^{-x} = \frac{\pi}{2}$. **1**

(iii) Hence deduce that $f(x) = \tan^{-1} e^x - \frac{\pi}{4}$ is an odd function. **1**

(iv) Sketch the graph of $y = \tan^{-1} e^x - \frac{\pi}{4}$ showing any asymptotes. **2**

QUESTION 4. [15 Marks] [START A NEW PAGE]

(a)



The parabola $x^2 = 4ay$ $(a > 0)$ touches the hyperbola $x^2 - \frac{y^2}{3} = 1$ at the points P and Q.

(i) Copy the diagram showing clearly for the hyperbola the intercepts made on the x-axis, the coordinates of the foci, the directrices with their equations and the equations of the asymptotes. **4**

(ii) Find the value of a and the coordinates of P and Q. **3**

(b) $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ are two variable points on the rectangular hyperbola $xy = c^2$ which move so that the points P, Q and $S\left(c\sqrt{2}, c\sqrt{2}\right)$ are always collinear. The tangents to the hyperbola at P and Q intersect at the point R.

(i) Show that the tangent to the hyperbola $xy = c^2$ at the point $T\left(ct, \frac{c}{t}\right)$ has equation $x + t^2y = 2ct$. **2**

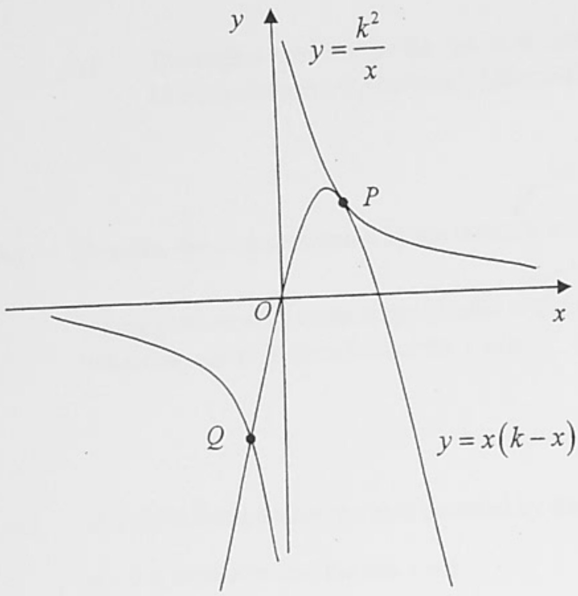
(ii) Hence show that R has coordinates $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$. **2**

(iii) Show that $p + q = \sqrt{2}\left(1 + pq\right)$. **2**

(iv) Hence find the equation of the locus of R. **2**

QUESTION 5. [15 Marks] [START A NEW PAGE]

(a)



The curves $y=x(k-x)$ and $y=\frac{k^2}{x}$, where $k>0$, touch at the point P and intersect at the point Q.

$x(k-x) = \frac{k^2}{x}$
$x^2(k-x) = k^2$
$x^2k - x^3 = k^2$
$x^3 - kx^2 + k^2 = 0$

(i) Deduce that the equation $x^3-kx^2+k^2=0$ has real roots α, α, β for some $\alpha \neq \beta$. **2**

(ii) Find the exact values of k, α and β. **4**

(b) (i) Show that $1-(\cos n\theta + i\sin n\theta) = -2i\sin\frac{n\theta}{2}\left(\cos\frac{n\theta}{2}+i\sin\frac{n\theta}{2}\right)$. **2**

(ii) Find the sum of the series $z+z^2+z^3+\ldots+z^n$ for $z\neq 1$. **1**

(iii) If $z=\cos\theta+i\sin\theta$, show that for $\sin\frac{\theta}{2}\neq 0$,

$$\cos\theta+\cos 2\theta+\cos 3\theta+\ldots+\cos n\theta = \frac{\sin\frac{n\theta}{2}\cos\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}}.$$ **4**

(iv) Hence solve the equation $\cos\theta+\cos 2\theta+\cos 3\theta=0, \quad 0\leq\theta\leq 2\pi$. **2**

QUESTION 6. [15 Marks] [START A NEW PAGE]

(a) (i) On the same axes draw the curves $y = x + 1$ and $y = (x-1)^2$ **2**

(ii) The region enclosed by the curves is rotated about the y – axis. Use the method of cylindrical shells to find the volume of the solid formed. **3**

(b) Consider the region bounded by the curve $y = \sqrt{x}$, the x axis and the line $x = 1$.

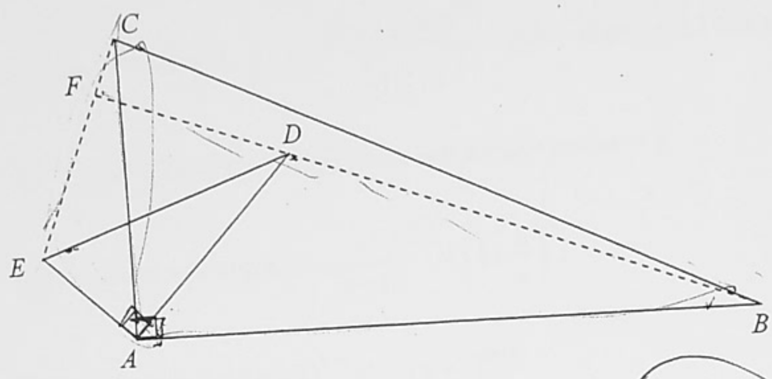
Use cylindrical shells with strips parallel to the y axis to find the volume of the solid formed when this region is rotated about the y axis. **5**

(c) A solid is formed when the area bounded by the curve, $y = \frac{1}{2}\sqrt{x-2}$, the x axis and the line $x = 6$ is rotated around the line $x = 6$.

By taking slices perpendicular to the line $x = 6$, find the volume of the solid formed. **5**

QUESTION 7. [15 Marks] [START A NEW PAGE]

(a)



Triangles ABC and ADE are each right angled at A, and $\Delta ABC ||| \Delta ADE$. BD produced meets CE at F.

(i) Copy the diagram.

(ii) Show that $\Delta BDA ||| \Delta CEA$. **3**

(iii) Hence show that show that $ADFE$ is a cyclic quadrilateral. **2**

(iv) Deduce that BF is perpendicular to CE. **2**

(b) A sequence T_n is given by $T_1 = 1$ and $T_n = \sqrt{3 + 2T_{n-1}}$ for $n = 2, 3, 4, \ldots$.

(i) Use Mathematical Induction to show that $T_n < 3$ for all positive integers $n \geq 1$. **3**

(ii) Hence show that $T_{n+1} > T_n$ for all positive integers $n \geq 1$. **2**

(iii) Show that $T_{n+2} - T_{n+1} = \dfrac{T_{n+2}^{\ 2} - T_{n+1}^{\ 2}}{T_{n+2} + T_{n+1}}$. Hence show that $T_{n+2} - T_{n+1} < T_{n+1} - T_n$ for all positive integers $n \geq 1$. **3**

QUESTION 8. [15 Marks] [START A NEW PAGE]

(a) Consider the function $f(x) = \dfrac{(n+1+x)^{n+1}}{(n+x)^n}$, $x \geq 0$ where $n \geq 1$ is a fixed positive integer.

(i) Show that for $x > 0$, $f(x)$ is an increasing function of x. **2**

(ii) Hence show that $\left(1+\dfrac{x}{n+1}\right)^{n+1} > \left(1+\dfrac{x}{n}\right)^n$. **2**

(iii) Deduce that $(n+2)^{n+1} n^n > (n+1)^{2n+1}$. **1**

(b) 6 envelopes are arranged in a straight line and numbered 1 to 6. Find the number of ways in which 6 different letters can be arranged one in each envelope

(i) if a given letter A is to be envelope 2. **1**

(ii) if letter A is in neither envelope 1 nor envelope 2, and letter B is not in envelope 2. **2**

(iii) if letter A is in neither envelope 1 and letter B is not in envelope 2. **2**

(c) (i) For positive real numbers a, b show that $a^2 + b^2 \geq 2ab$. **1**

(ii) Hence show for positive real numbers a, b, c, d
$3(a^2+b^2+c^2+d^2) \geq 2(ab+ac+ad+bc+bd+cd)$ **2**

(iii) Hence show that if a, b, c, d are positive real numbers such that $a+b+c+d=1$ then $ab+ac+ad+bc+bd+cd \leq \dfrac{3}{8}$. **2**