CARINGBAH HIGH SCHOOL

2011

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

Reading time - 5 minutes

Working time - 3 hours

Write using black or blue pen.

Board-approved calculators may be used.

A table of standard integrals is provided at the back of this paper.

Total marks - 120

Attempt Questions 1 - 8

All questions of equal value.

All necessary working should be shown in every question.

Question 1 (15 marks) **Marks**

(a) Find $\int \frac{dx}{(2x+1)^3}$ **2**

(b) Using integration by parts find the exact value of $\int_0^{\frac{1}{2}} \cos^{-1}x \, dx$. **3**

(c) Use the substitution $u = x - 1$ to find $\int \frac{x}{\sqrt{x-1}} dx$ **3**

(d) Use the substitution $t = \tan\frac{x}{2}$ to find the exact value of **4**

$$\int_0^{\frac{\pi}{2}} \frac{1}{\sin x + 2} dx$$

(e) Evaluate $\int_0^1 \frac{5}{(2x+1)(2-x)} dx$ **3**

Question 2 (15 marks) Start a new page.

(a) Find the complex square roots of $7+6\sqrt{2}\,i$ giving your answer in the form $x+iy$ where x and y are real. **2**

(b) If $z=3+i$ find $\dfrac{i}{z}$ in the form $x+iy$. **2**

(c) Let $z_1=3+6i$ and $z_2=-3-6i$.

Show that the locus specified by $|z-z_1| = 2|z-z_2|$ is a circle. **2**
Give its centre and radius.

(d) (i) Express $-1+i$ in modulus-argument form. **1**

(ii) Express $(-1+i)^6$ in the form $x+iy$. **2**

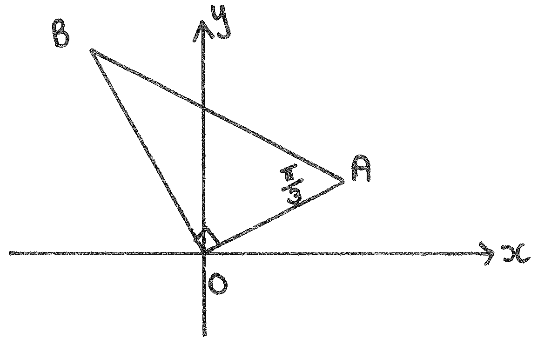
(e) Sketch the locus of z satisfying:

(i) $\arg(z+2) = \dfrac{\pi}{4}$. **2**

(ii) $\text{Re}(z) = |z|$. **2**

Question 2 continues on page 4

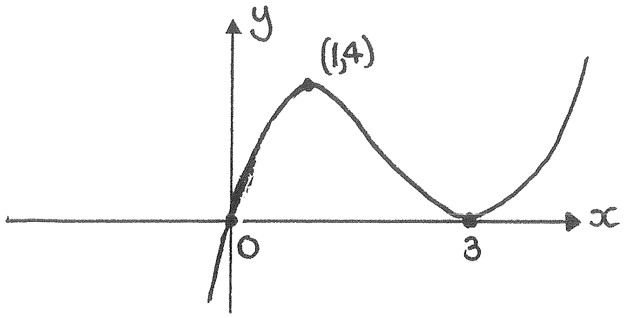
(f) In the diagram below, the points A and B correspond to the complex numbers z and w respectively. $\angle AOB$ is a right angle and $\angle BAO = \frac{\pi}{3}$.



Show that $3z^2 + w^2 = 0$. **2**

Question 3 (15 marks) Start a new page.

(a) The function defined by $g(x) = x(x-3)^2$ is drawn below.



Draw separate, one-third page sketches of :

(i) $y = g(|x|)$ **1**

(ii) $y = \dfrac{1}{g(x)}$ **2**

(iii) $y = \sqrt{g(x)}$ **2**

(iv) $y = \tan^{-1}[g(x)]$ **2**

(b) For the curve $x^2 + y^2 + xy - 4 = 0$:

(i) Find the x and y intercepts. **1**

(ii) Using implicit differentiation show that $\dfrac{dy}{dx} = -\dfrac{2x+y}{x+2y}$ **2**

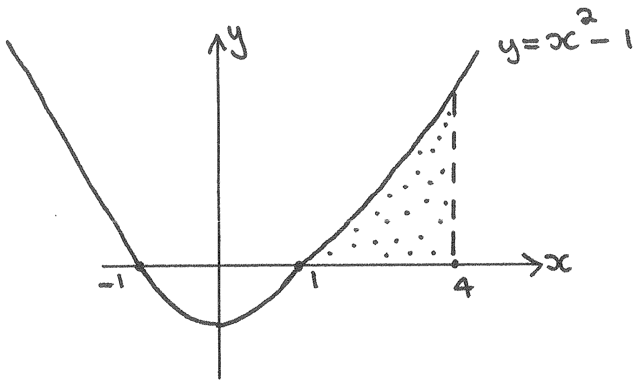
(iii) Find any stationary points on the curve. **2**

(iv) Deduce that the curve has vertical tangents at the points where $x = \pm\dfrac{4}{\sqrt{3}}$. **2**

(v) Sketch the curve $x^2 + y^2 + xy - 4 = 0$. **1**

Question 4 (15 marks) Start a new page.

(a) **4**



The area bounded by the curve $y = x^2 - 1$, the x-axis and the line $x = 4$, as shown in the diagram, is rotated about the y-axis to form a solid. Use the method of cylindrical shells to find the volume of the solid.

(b) Sketch the graph of the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$ showing the intercepts on the axes, the coordinates of the foci and the equations of the directrices. **4**

(c) The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with $a > b > 0$ has eccentricity e.

(i) Show that the line through the focus $S(ae, 0)$ which is perpendicular to the asymptote $y = \frac{bx}{a}$ has equation $ax + by - a^2e = 0$. **1**

(ii) Show that this line meets the asymptote at a point on the corresponding directrix. **3**

(d) Consider the polynomial $P(x) = x^3 - x^2 + x + 39$.

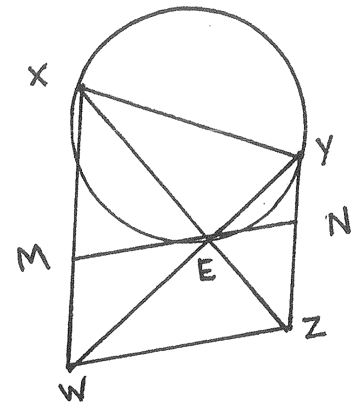
(i) Find the rational zero of $P(x)$. **1**

(ii) Find the complex zeros of $P(x)$. **2**

Question 5 (15 marks) Start a new page.

(a) In the diagram below, $XYZW$ is a cyclic quadrilateral whose diagonals intersect at E. A circle is drawn through X, Y and E. MN is a tangent to this circle at E with M and N lying on XW and YZ respectively. 3

Copy this diagram.



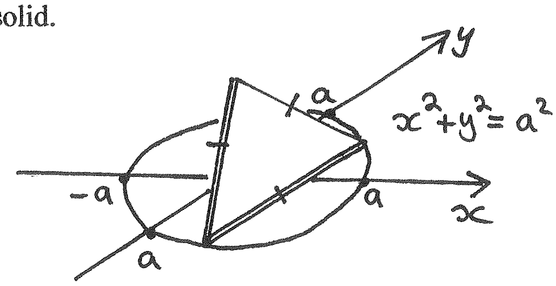
Prove that MN is parallel to WZ.

(b) Suppose that p and q are real numbers.

(i) Show that $pq \leq \frac{p^2+q^2}{2}$. 2

(ii) Hence show that for x and y real numbers $\frac{1}{xy} \leq \frac{x^2+y^2}{2x^2y^2}$. 2

(c) The base of a certain solid is the circle $x^2 + y^2 = a^2$. 3
Each cross-section of the solid is an equilateral triangle parallel to the y-axis with one side lying on the circle, as shown in the diagram.
Find the volume of the solid.



Question 5 continues on page 8

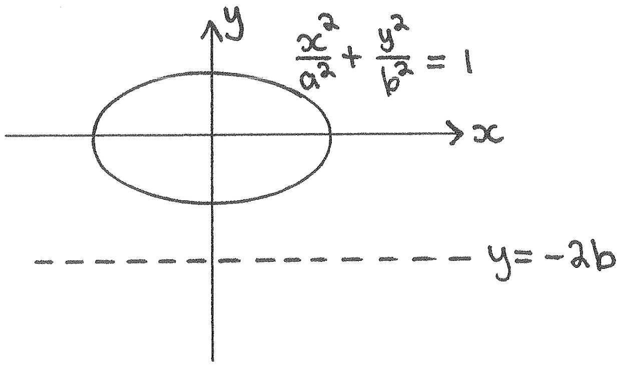
(d) The complex cube roots of unity ω , ω^2 are two of the roots of $P(x) = x^3 + px^2 + qx + r$. **3**

Show that $p = q = r + 1$.

(e) Resolve $\dfrac{1}{(x-3)(x^2+1)}$ into partial fractions over the field of real numbers. **2**

Question 6 (15 marks) Start a new page.

(a) The area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is rotated about the line $y = -2b$. A strip of thickness δx perpendicular to the axis of rotation sweeps out a slice whose cross-section is an annulus.



(i) Show that this slice has a volume of $\delta V = 8\pi b y \delta x$. **2**

(ii) Hence find the volume of the solid which is formed. **3**

(b) (i) If $I_n = \int_{-1}^{0} x^n (1+x)^{\frac{1}{2}} \, dx$ show that $I_n = -\frac{2n}{2n+3} I_{n-1}$. **3**

(ii) Hence evaluate I_3. **2**

(c) By differentiating both sides of the formula: **3**

$1 + x + x^2 + x^3 + \ldots + x^n = \frac{x^{n+1} - 1}{x - 1}$ find an expression for:

$1 + 2\times 2 + 3\times 4 + 4\times 8 + \ldots + n2^{n-1}$.

(d) Given that $1-2i$ is a zero of the polynomial $p(x) = x^3 - 5x^2 + 11x - 15$ factorise $p(x)$ over the field of complex numbers. **2**

Question 7 (15 marks) Start a new page.

(a) The normal at the point $P\left(cp, \dfrac{c}{p}\right)$ on the hyperbola $xy = c^2$ meets the x-axis at Q. M is the midpoint of PQ.

(i) Show that the normal at P has the equation $p^3x - py = c(p^4 - 1)$. **2**

(ii) Show that M has coordinates $\left(\dfrac{c(2p^4 - 1)}{2p^3}, \dfrac{c}{2p}\right)$ **3**

(iii) Hence or otherwise, find the equation of the locus of M. **2**

(b) The numbers a, b and c are said to be in harmonic progression if their reciprocals $\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$ are in arithmetic progression, and b is then said to be the harmonic mean of a and c.

(i) Show that the numbers $6, 8$ and 12 are in harmonic progression. **1**

(ii) Show that the harmonic mean of a and c is $\dfrac{2ac}{a+c}$. **2**

(iii) If $a > 0, c > 0$ show that the geometric mean $\sqrt{ac}$ is greater than or equal to the harmonic mean $\dfrac{2ac}{a+c}$. **2**

(c) (i) Sketch $y = x^2 - 2x - 1$ showing the x-intercepts. **1**

(ii) Using mathematical induction and part (i) prove that $2^n > n^2$ for all integers $n \geq 5$. **2**

Question 8 (15 marks) Start a new page.

(a) The roots of the equation $x^3+px+m=0$ where $m \neq 0$ are α, β and δ. 2
Find an equation expressed in the form $ax^3+bx^2+cx+d=0$ whose roots are α^{-2}, β^{-2} and δ^{-2}.

(b) The vertices of a quadrilateral *ABCD* lie on a circle radius *r*. The angles subtended at the centre of the circle by sides *AB*, *BC*, *CD* and *DA* are respectively in an arithmetic progression with first term *a* and common difference *d*. (i.e. *AB* subtends an angle of *a*).

(i) Show that $2a + 3d = \pi$ and interpret this result geometrically. 2

(ii) Show that the area of the quadrilateral *ABCD* is $2r^2 \cos d \cos\frac{d}{2}$. 3

[If required you may use the result: $\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$]

(c) Let $p = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$.
The complex number $\alpha = p + p^2 + p^4$ is a root of the quadratic equation $x^2 + ax + b = 0$, where *a* and *b* are real.

(i) Prove that $1 + p + p^2 + ... + p^6 = 0$. 2

(ii) The second root of the quadratic equation is β. Justifying your answer, express β in terms of positive powers of *p*. 2

(iii) Find the values of the coefficients *a* and *b*. 2

(iv) Deduce that $-\sin\frac{\pi}{7} + \sin\frac{2\pi}{7} + \sin\frac{3\pi}{7} = \frac{\sqrt{7}}{2}$. 2

End of paper