2009

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

MATHEMATICS EXTENSION 2

General Instructions

- Reading Time – 5 minutes
- Working Time - 3 hours
- Write using blue or black pen
- Board approved calculators may be used
- Write your name on each page
- Each question is to be started on a new page.
- This examination paper must NOT be removed from the examination room

- There is a total of eight questions.
- Each question is worth 15 marks.
- Marks may be deducted for careless or badly arranged work.

Question 1 (15 marks) Start a NEW page. **Marks**

a) Find $\int \cos x \sin^4 x \, dx$. 1

b) Find $\int \frac{dx}{x^2 - 4x + 8}$. 2

c) Use the substitution $u = x - 2$ to find the exact value of $\int_1^3 x(x-2)^5 \, dx$. 3

d) i) Find the values of A, B and C so that 2

$$\frac{5}{(x^2+4)(x+1)} \equiv \frac{Ax+B}{x^2+4} + \frac{C}{x+1}.$$

ii) Hence find $\int \frac{5}{(x^2+4)(x+1)} \, dx$. 3

e) i) If $I_n = \int_1^e x(\ln x)^n \, dx$ for $n = 0,1,2,3,\ldots$ use integration by parts 2

to show that $I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1}$ for $n = 1,2,3,\ldots$

ii) Hence find the value of I_2 . 2

Question 2 (15 marks) Start a NEW page. **Marks**

a) If $z = 1 - i$, find

i) $\bar{z}$ 1

ii) $|z|$ 1

iii) $\arg z$ 1

iv) $\arg iz$ 1

v) z^6 in $a + ib$ form. 2

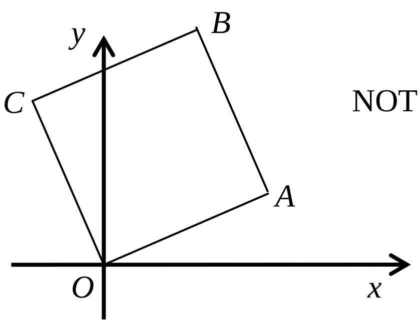
b) Express $\dfrac{i^5(1-i)}{2+i}$ in the form $a + ib$ where a and b are real. 3

c) Graph the region in the Argand diagram which simultaneously satisfies 3

$1 \le |z - i| \le 2$ and $\text{Im}\, z \ge 0$.

d) In the Argand diagram A represents the point $z_1 = \sqrt{3} + i$, and O is the origin.

Given that $OABC$ is a square:



NOT TO SCALE

i) Find the complex number (z_3) represented by C. 1

ii) Find the complex number (z_2) represented by B. 2

Question 3 (15 marks) Start a NEW page. **Marks**

a) i) Without using calculus, draw a good size neat sketch of $y=(x+1)^2(1-x)$. 2

ii) On a separate diagram using the same scale as above, and also without calculus, sketch $y^2=(x+1)^2(1-x)$, paying close attention to the shape of the curve as 'y' approaches zero. 2

b) Neatly sketch each of the following on separate axes for $0 \le x \le 2\pi$.

i) $y=\sin^2 x$. 1

ii) $y=\left|\sin x\right|$. 1

iii) $y=\sqrt{\sin x}$. 1

iv) $y=\dfrac{1}{\sin x}$. 1

v) $y=\dfrac{\left|\sin x\right|}{\sin x}$. 1

vi) $y=e^{\sin x}$. 2

c) A plane curve is defined by the equation $x^2+2xy+y^5=4$. 4

The curve has a horizontal tangent at the point $P(X,Y)$.

By using implicit differentiation or otherwise, show that X is the unique solution to the equation $X^5+X^2+4=0$.

[Do not solve this equation]

Question 4 (15 marks) Start a NEW page. **Marks**

a) Given the hyperbola $16x^2 - 9y^2 = 144$, find:

i) the length of the transverse axis 1

ii) the eccentricity 1

iii) the coordinates of the foci 1

iv) the equations of the directrices 1

v) the equations of the asymptotes. 1

b) If α, β and γ are the roots of the equation $x^3 + 6x + 1 = 0$, find the polynomial equation whose roots are $\alpha\beta, \beta\gamma$ and $\alpha\gamma$. 3

c) Consider the equation $x^4 - 5x^3 + 7x^2 + 3x - 10 = 0$.

i) Given that $2 - i$ is a root of the equation explain why $2 + i$ is also a root. 1

ii) Find the other roots of the equation. 3

d) The area enclosed between the curves $y = \sqrt{x}$ and $y = x^2$ is rotated about y – axis through one complete revolution. Use the method of cylindrical shells to find the volume of the solid that is generated. 3

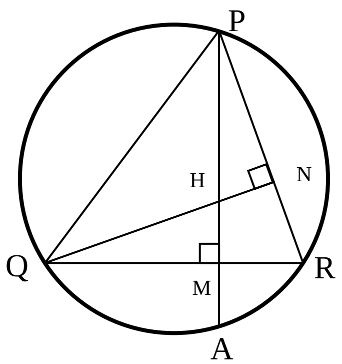
Question 5 (15 marks) Start a NEW page. **Marks**

a) Solve for x: $2^{3x+1} = 5^{x+1}$, correct to 3 significant figures. 2

b) i) Prove the result $\int_{-a}^{a} f(x)\, dx = \int_{0}^{a} \left[f(x) + f(-x) \right] dx$ 2

ii) Hence use the above result to evaluate $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\sin x}\, dx$ 3

c) The altitudes PM and QN of an acute angled triangle PQR meet at H. PM produced cuts the circle PQR at A. [A larger diagram is included, use it and submit it with your solutions]



i) Explain why PQMN is a cyclic quadrilateral. 1

ii) Prove that HM = MA. 3

d) The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. The tangent at P cuts the y- axis at B and M is the foot of the perpendicular from P to the y-axis.

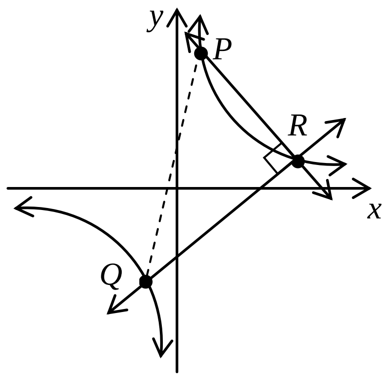
i) Show that the equation of the tangent to the ellipse at the point P is given by $\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$ 2

ii) Show that $OM \bullet OB = b^2$, where O is the origin. 2

Question 6 (15 marks) Start a NEW page. **Marks**

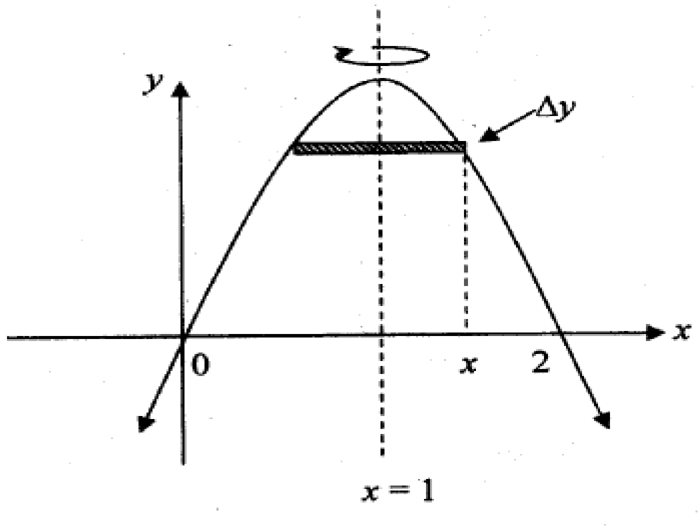
a) The points $P\left(cp, \dfrac{c}{p}\right)$ and $Q\left(cq, \dfrac{c}{q}\right)$ lie on the rectangular hyperbola $xy = c^2$. 4

The chord PQ subtends a right angle at another point $R\left(cr, \dfrac{c}{r}\right)$ on the hyperbola.



Show that the normal at R is parallel to PQ.

b) The area bounded by the curve $y = 2x - x^2$ and the x-axis is rotated through 180° about the line $x = 1$.



i) Show that the volume, ΔV, of a representative horizontal slice of width Δy is given by $\boldsymbol{\Delta V = \pi(x-1)^2\Delta y}$. 2

ii) Hence show that the volume of the solid of revolution is given by 2

$$\boldsymbol{V = \lim_{\Delta y \to 0} \sum_{y=0}^{1} \pi(1-y)\Delta y}$$

iii) Hence, find the volume of the solid of revolution. 2

Question 6 continued on page 8

Question 6 continued **Marks**

c) i) Show that $a^2 + b^2 > 2ab$, where a and b are distinct positive real numbers. 1

ii) Hence show that $a^2 + b^2 + c^2 > ab + bc + ca$, where a, b and c are distinct positive real numbers. 2

iii) Hence, or otherwise prove that $\dfrac{a^2b^2 + b^2c^2 + c^2a^2}{a+b+c} > abc$, where a, b and c are distinct positive real numbers. 2

End of Question 6

Question 7 (15 marks) Start a NEW page. **Marks**

a) Given that $P(x) = 3x^3 - 11x^2 + 8x + 4$ has a double root, fully factorise $P(x)$. 3

b) Show that $\tan^{-1} x > x - \frac{1}{3}x^3$ for all values of $x > 0$. 3

c) The acceleration of a particle which is moving along the x-axis is given by

$$\frac{d^2x}{dt^2} = 2x^3 - 10x.$$

i) If the particle starts at the origin with velocity u show that its velocity v is given by $v^2 - u^2 = x^4 - 10x^2$. 2

ii) If $u = 3$ show that the particle oscillates within the interval $-1 \leq x \leq 1$. 4

iii) Is the motion referred to in (ii) an example of simple harmonic motion? Give a clear reason for your answer. 1

iv) If $u = 6$, carefully describe the motion. 2

Question 8 (15 marks) Start a NEW page. **Marks**

a) Let ω be one of the non-real cube roots of unity.

i) Show that $1 + \omega + \omega^2 = 0$. 1

ii) Hence find the value of $(2-\omega)(2-\omega^2)(2-\omega^4)(2-\omega^5)$. 2

b) i) By using the expansion for $\cos(A+B)$, show that 2

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

ii) By using the substitution $x = 2\cos\theta$, solve the equation $x^3 - 3x = \sqrt{2}$. 3

iii) Hence explain why $\cos\left(\frac{\pi}{12}\right) + \cos\left(\frac{7\pi}{12}\right) + \cos\left(\frac{9\pi}{12}\right) = 0$ 1

c) The equation $x^3 - 3px + q = 0$, where $p > 0, q \neq 0$ are both real, has three distinct, non- zero real roots.

i) Show that the graph of $y = x^3 - 3px + q$ has a relative maximum value of $q + 2p\sqrt{p}$ and a relative minimum of $q - 2p\sqrt{p}$. 3

ii) Hence show giving reasons that $q^2 < 4p^3$. 3

END OF EXAM