BURWOOD GIRLS HIGH SCHOOL

2005
TRIAL HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics
Extension 2

9/8/05

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using a blue or black pen
- Board approved calculators may be used
- A table of standard integrals is provided with this paper
- All necessary working should be shown for every question
- All questions are out of 15.

Marks

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\displaystyle\int_0^1 \frac{2}{\sqrt{1+3x}}\,dx$. **2**

(b) By using *integration by parts*, find $\displaystyle\int x^2 \ln 2x\,dx$. **2**

(c) Evaluate $\displaystyle\int_0^{\frac{\pi}{6}} \sin^3 2x\,dx$. **3**

(d) Using $t = \tan\frac{x}{2}$, find $\displaystyle\int \frac{dx}{1+\sin x}$. **4**

(e) (i) Find real constants A, B and C such that **2**

$$\frac{x+4}{x(x^2+4)} \equiv \frac{A}{x} + \frac{Bx+C}{x^2+4}.$$

(ii) Hence find $\displaystyle\int \frac{x+4}{x(x^2+4)}\,dx$. **2**

$\frac{80}{768}$ $\frac{10}{96}$ $\frac{1}{64} - \frac{1}{384}$ $\frac{384-64}{24576}$ $\frac{320}{3072}$

$\sin^3 2x$ $(t+1)^{-1}$ $\frac{1}{t+1}$ $-\frac{1}{(t+1)^2}$

$\sin 2x(1-\cos^2 2x)$ $\frac{5}{48}$

$\sin 2x - \sin 2x\cos^2 2x$

$\sin 2x - -\frac{1}{384}$

$u = \sin 2x$

$du = 2\cos 2x\,dx$

$u = \sin 2x$

$du = 2\cos 2x\,dx$

$u(1-\cos^2 2x)$ $\frac{1}{16}$ $\frac{1}{4}$

$\sin^2 2x$

$\sin 2x = 2\sin x\cos x$

$(2\sin x\cos x)^3$

$8\sin^3 x\cos^3 x$

$8\sin^3 x(1-\sin^2 x)\cos x$

$u^2 = \sin^2 2x$

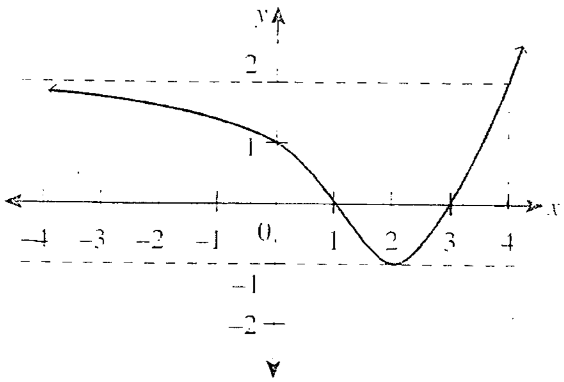
$u^2 = 1-\cos^2 2x$

$\therefore \cos^2 2x = 1-u^2$

$u^2 = \sin^2 2x$

Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The diagram below shows the graph of $y = f(x)$.



On separate diagrams, sketch the following, showing essential features.

(i) $y = \dfrac{1}{f(x)}$ 2

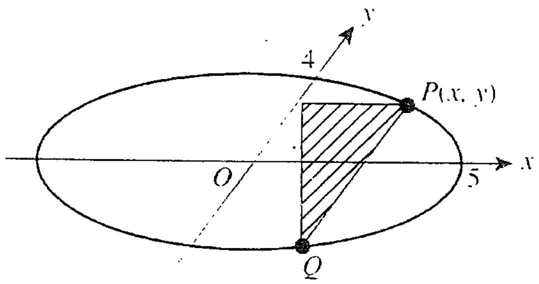
(ii) $y = f(x+2)$ 2

(iii) $y^2 = f(x)$ 2

(iv) $y = \ln f(x)$ 2

(b) Find the equation of the tangent to the curve $x^3 + y^3 - 3xy = 3$ at the point $(1, 2)$. 3

(c)



x - 1 - 3y + t

The base of a certain solid is the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$.
Every cross-section perpendicular to the x-axis is an equilateral triangle. The shaded cross-section is thus an equilateral triangle with base PQ.

(i) Show that the shaded cross-sectional area is given by 1

$$A = \sqrt{3}y^2.$$

V = πab

(ii) Hence find the cross-sectional area as a function of x. 1

(iii) Find the volume of the solid. 2

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Show that $(1+i)^3 = 2(i-1)$. **1**

(b) By evaluating, or otherwise, show that $\dfrac{1+2i}{3-4i} + \dfrac{2-i}{5i}$ is a real number. **2**

(c) Draw on one Argand diagram the three loci:

(i) $|z-i| = 1$, **3**

$\arg(z-i) = \dfrac{\pi}{3}$,

$|z-i| = |z-3i|$.

(ii) Hence calculate the area of the intersection of the three loci: **1**

$$|z-i| \le 1,\quad \frac{\pi}{3} \le \arg(z-i) \le \frac{\pi}{2} \quad \text{and} \quad |z-i| \le |z-3i|.$$

(d) Let $p(z) = 2z^3 - 5z^2 + qz - 5$, where q is a real number.

(i) If $p(1-2i) = 0$, solve $p(z) = 0$. **2**

(ii) Hence determine the value of q if $p(1-2i) = 0$. **1**

(e) Let z be a complex number such that $z \neq 0$ and $z \neq 1$, and

$$\frac{z}{\bar{z}} = -\frac{z-1}{\overline{z-1}}$$

$\dfrac{a+ib}{a-ib} = -\overline{a}$

$\arg z - \arg \bar{z}$

(i) Show that for any non-zero complex number z, **1**

$$\arg\left(\frac{z}{\bar{z}}\right) = 2\arg z.$$

(ii) Let z be a complex number such that $z \neq 0$ and $z \neq 1$, and **2**

$$\frac{z}{\bar{z}} = -\frac{z-1}{\overline{z-1}}.$$

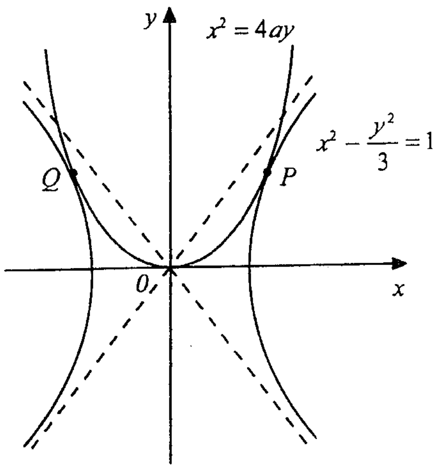
Show that $\arg z = \arg(z-1) + \dfrac{\pi}{2}$ or $\arg z = \arg(z-1) - \dfrac{\pi}{2}$.

(iii) Hence sketch the locus of all points z that satisfy **2**

$$\frac{z}{\bar{z}} = -\frac{z-1}{\overline{z-1}}.$$

Question 4 (15 Marks) Use a separate writing booklet. **Marks**

(a)



The parabola $x^2 = 4ay$ $(a > 0)$ touches the hyperbola $x^2 - \frac{y^2}{3} = 1$ at the points P and Q.

(i) Copy the diagram showing clearly for the hyperbola the intercepts made on the x axis, the coordinates of the foci, the directrices with their equations and the equations of the asymptotes. 4

(ii) Find the value of a and the coordinates of P and Q. 3

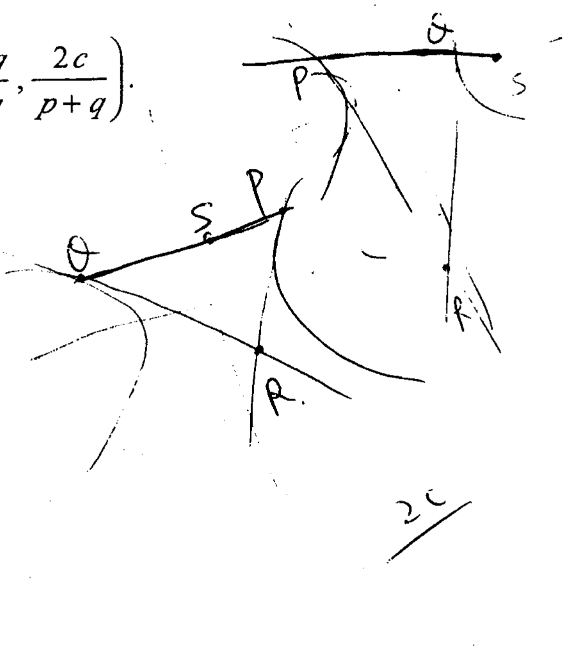
(b) $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ are two variable points on the rectangular hyperbola $xy = c^2$ which move so that the points P, Q and $S\left(c\sqrt{2}, c\sqrt{2}\right)$ are always collinear. The tangents to the hyperbola at P and Q intersect at the point R.

(i) Show that the tangent to the hyperbola $xy = c^2$ at the point $T\left(ct, \frac{c}{t}\right)$ has equation $x + t^2 y = 2ct$. 2

(ii) Hence show that R has coordinates $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$. 2

(iii) Show that $p + q = \sqrt{2}\left(1 + pq\right)$. 2

(iv) Hence find the equation of the locus of R. 2



Question 5 (15 Marks) Use a separate writing booklet.

(a) The equation $x^3 + 2x + 1 = 0$ has roots α, β, and γ. **Marks**

(i) Find the monic cubic equation with roots α^{-1}, β^{-1} and γ^{-1} 2

(ii) Find the monic cubic equation with roots α^{-2}, β^{-2} and γ^{-2} 2

(b) (i) Sketch the region containing all points that simultaneously satisfy the following: 2

$$x \le 1,\ y \ge 1, \text{ and } y \le e^x.$$

(ii) The region is part (i) is rotated through one complete revolution about the x-axis. Use the method of cylindrical shells to show that the volume of the resulting solid is given by $\frac{\pi}{2}(e^2 - 3)$ cubic units. 4

(c) A conical pendulum consists of a mass of M kg hanging at the end of a light string of length 1 metre attached from a fixed point O.

The mass rotates in a circle and moves with a period of S seconds.

The string makes a constant angle of θ to the vertical.

$ST = \frac{2\pi}{\omega}$ ytent: $= r\omega^2$

(i) Use a sketch to illustrate the forces acting on the mass 1

ω $\cancel{\cos}$

(ii) By resolving the forces acting on the mass, show that $S = 2\pi\sqrt{\dfrac{\cos\theta}{g}}$, 2

where g is the acceleration due to gravity. $\frac{1}{\cot\theta}$ SNE: F

$\frac{1}{\sec\theta}$

(iii) The string can just support a stationary mass of $5M$ kg hanging vertically. 2

Find the smallest period that the conical pendulum can have, also leaving your answer in terms of g.

Marks

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that $\displaystyle\int_0^a f(x)dx = \int_0^a f(a-x)dx$. **2**

(ii) Deduce that $\displaystyle\int_0^{\frac{\pi}{4}} \frac{1-\sin 2x}{1+\sin 2x}\,dx = \int_0^{\frac{\pi}{4}} \tan^2 x\,dx$ · **2**

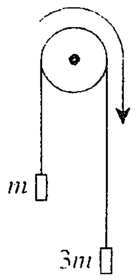
(iii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{1-\sin 2x}{1+\sin 2x}\,dx$ · **2**

(b) Particles of mass $3m$ kg and m kg are connected by a light inextensible string which passes over a smooth fixed pulley, the string hanging vertically on each side.

The particles are released from rest and move under gravity.

The air resistance on each particle is kv newtons when the speed of the particle is v m s^{-1}.

Take the positive direction of motion as indicated by the arrow in the diagram below.



Let the tension in the string acting on the masses have a magnitude of T newtons.

(i) By resolving the forces on both particles, show that the equation of motion of the system is given by **2**

$$\frac{dv}{dt} = \frac{mg - kv}{2m}.$$

(ii) Hence find the terminal velocity of the system, stating your answer in terms of m, g and k. **1**

(iii) Prove that the time elapsed since the beginning of the motion is given by **3**

$$t = \frac{2m}{k}\log_e\left(\frac{mg}{mg - kv}\right).$$

(iv) If the bodies have attained a speed equal to half the terminal speed, show that the time elapsed is equal to **3**

$$\frac{V}{g}\log_e 4,$$

where V is the terminal speed.

$\cos^2 x - \sin^2 x$

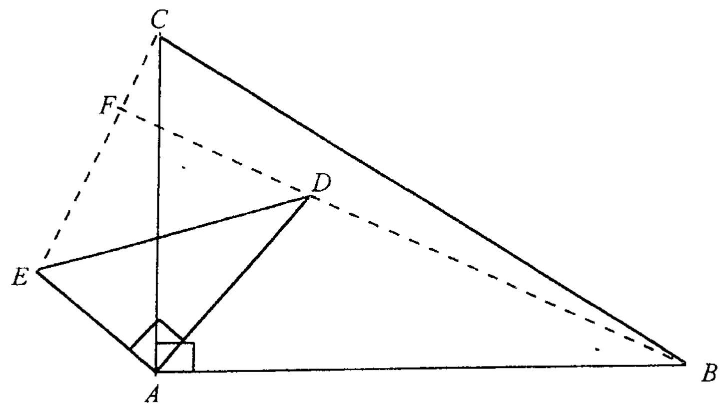
$\cos 2x =$

$1 - 2\sin^2 x$

$\sec^2 = 1 + \tan^2 x$

Question 7 (15 Marks) Use a separate writing booklet. **Marks**

(a)



Triangles ABC and ADE are each right angled at A, and $\Delta ABC \,|||\, \Delta ADE$.
BD produced meets CE at F.

(i) Copy the diagram.

(ii) Show that $\Delta BDA \,|||\, \Delta CEA$. 3

(iii) Hence show that $ADFE$ is a cyclic quadrilateral. 2

(iv) Deduce that $BF \perp CE$. 2

(b) A sequence T_n is given by $T_1 = 1$ and $T_n = \sqrt{3 + 2T_{n-1}}$ for $n = 2, 3, 4, \ldots$

(i) Use Mathematical Induction to show that $T_n < 3$ for all positive integers $n \geq 1$. 3

(ii) Hence show that $T_{n+1} > T_n$ for all positive integers $n \geq 1$. 2

(iii) Show that $T_{n+2} - T_{n+1} = \dfrac{T_{n+2}^{\,2} - T_{n+1}^{\,2}}{T_{n+2} + T_{n+1}}$. Hence show that $T_{n+2} - T_{n+1} < T_{n+1} - T_n$ 3
for all positive integers $n \geq 1$.

Question 8 (15 Marks) Use a separate writing booklet. **Marks**

(a)(i) Show that for real numbers a and b, $a^2 + b^2 \geq 2ab$. 1

$a^2bc + ab^2c + abc^2$

(ii) Hence show that for real numbers a, b and c,
$a^2 + b^2 + c^2 \geq ab + bc + ca$ and $a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a + b + c)$. 3

(iii) The equation $x^3 + px^2 + qx + r = 0$, where p, q and r are real, has three real roots a, b and c. Show that $q^2 \geq 3pr$. 3

(b) (i) With the aid of a diagram show that $\int_1^{\sqrt{u}} \frac{dx}{x} < \sqrt{u} - 1$ for $u > 1$. 1

(ii) Hence show that $0 < \ln u < 2(\sqrt{u} - 1)$, for $u > 1$. 2

(iii) Hence show that $\frac{\log u}{u} \to 0$, as $u \to \infty$. 1

(c) Let p and q be non-zero real numbers such that $q(1 + p + q) < 0$.

(i) Show that the equation $x^2 + px + q = 0$ has exactly one real root in the interval $0 < x < 1$. 1

(ii) Show that the equation $x^2 + px + q = 0$ has two distinct real roots, one positive and one negative. 3

$(ab + ac + bc)(ab + ac + bc)$

$a^2b^2 + a^2bc + ab^2c + a^2bc + a^2c^2 + abc^2 + ab^2c + abc^2$

$2a^2bc + 2ab^2c +$

qp