Formula for π in terms of binomial coefficients

by Derek Buchanan

We saw in the 2014 Mathematics Extension 2 HSC exam [1] that

$$\tfrac{\pi}{4} = \sum_{r=1}^{\infty} \tfrac{(-1)^{r-1}}{2r-1} \quad (*)$$

and in the 1957 Mathematics I Honours Leaving Certificate exam [2] that

$$\ln 2 = \sum_{r=1}^{\infty} \tfrac{(-1)^{r-1}}{r} \quad (\dagger)$$

We combine these formulae to produce a new formula for π in terms of binomial coefficients (all of them being the triangular numbers), namely the

Proposition. $2 + \binom{2}{2}^{-1} + \binom{3}{2}^{-1} - \binom{4}{2}^{-1} - \binom{5}{2}^{-1} + \binom{6}{2}^{-1} + \binom{7}{2}^{-1} - \cdots = \pi$

Proof. Suppose $A = \sum_{r=1}^{\infty}(-1)^{r-1}\binom{2r}{2}^{-1}$ and $B = \sum_{r=1}^{\infty}(-1)^{r-1}\binom{2r+1}{2}^{-1}$. Then

$$\begin{aligned}
A &= \sum_{r=1}^{\infty} 2(-1)^{r-1}(\tfrac{1}{2r-1} - \tfrac{1}{2r}) \\
&= 2\sum_{r=1}^{\infty} \tfrac{(-1)^{r-1}}{2r-1} - 2\sum_{r=1}^{\infty} \tfrac{(-1)^{r-1}}{2r} \\
&= 2 \cdot \tfrac{\pi}{4} - \sum_{r=1}^{\infty} \tfrac{(-1)^{r-1}}{r} \text{ from } (*) \\
&= \tfrac{\pi}{2} - \ln 2 \text{ from } (\dagger) \text{ and}
\end{aligned}$$

$$\begin{aligned}
B &= \sum_{r=1}^{\infty} 2(-1)^{r-1}(\tfrac{1}{2r} - \tfrac{1}{2r+1}) \\
&= \sum_{r=1}^{\infty} \tfrac{(-1)^{r-1}}{r} - 2\sum_{r=1}^{\infty} \tfrac{(-1)^{r-1}}{2r+1} \\
&= \ln 2 + 2\left(-1 + \sum_{r=1}^{\infty} \tfrac{(-1)^{r-1}}{2r-1}\right) \text{ from } (\dagger) \\
&= \ln 2 + 2(-1 + \tfrac{\pi}{4}) \text{ from } (*) \\
&= \ln 2 + \tfrac{\pi}{2} - 2.
\end{aligned}$$

As A and B are absolutely convergent,

$$\begin{aligned}
\therefore A + B + 2 &= 2 + \sum_{r=1}^{\infty}(-1)^{r-1}\left(\binom{2r}{2}^{-1} + \binom{2r+1}{2}^{-1}\right) \\
&= 2 + \tfrac{\pi}{2} - \ln 2 + \ln 2 + \tfrac{\pi}{2} - 2 \\
&= \pi
\end{aligned}$$

or in expanded form, $2 + \binom{2}{2}^{-1} + \binom{3}{2}^{-1} - \binom{4}{2}^{-1} - \binom{5}{2}^{-1} + \binom{6}{2}^{-1} + \binom{7}{2}^{-1} - \cdots = \pi$ □

Note: This formula was discovered in 2007 by J.C. Toloza [3] and the proof presented here is a vastly simplified version of his proof.

References

[1] https://educationstandards.nsw.edu.au/wps/portal/nesa/resources-archive/hsc-exam-papers-archive/mathematics_extension_2/mathematics-extension-2-2014-hsc-exam-pack-archive

[2] http://4unitmaths.com/lc1956-1962.pdf

[3] http://www.xtec.cat/~bfiguera/formulpi.htm