Beta Functions in Past Papers

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Here are 12 examples from 1945-2023 LC and HSC exams where you can use beta functions resulting in considerably shorter solutions than otherwise would be the case.

1945. Find $\int_0^{\frac{\pi}{2}} \cos^3 x \, dx$

1951. Find $\int_0^{a} \frac{x^{\frac{3}{2}}}{\sqrt{a-x}} \, dx$

1956. Find $\int_0^{t} x^2 (t-x)^{\frac{3}{2}} \, dx$

1962. Find $\int_0^{\frac{\pi}{2}} \sin^6 \theta \, d\theta$

1980. Find $\int_0^{1} x(1-x)^{20} \, dx$

1980. Find $\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

1985. Find $\int_0^{1} x(1-x)^{99} \, dx$

1987. Find $\int_0^{\frac{\pi}{2}} \cos^5 x \, dx$

1991. Find $\int_0^{\frac{\pi}{2}} \sin^4 x \, dx$

1995. Show that $\int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$ and

$\int_0^{\frac{\pi}{2}} \sin^{2n+1} x \, dx = \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdot \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot 1$

2020. Show that $\int_0^{1} x^n (1-x)^n \, dx = \frac{(n!)^2}{(2n+1)!}$

2023. Show that $\int_0^{1} x^n (1-x)^n \, dx = \frac{1}{2^{2n}} \int_0^{\frac{\pi}{2}} \sin^{2n+1} x \, dx$

Solutions are on the next 2 pages.

Solutions

1945 Leaving Certificate Mathematics I Honours Q10iiib

$\int_0^{\frac{\pi}{2}} \cos^3 x\, dx = \frac{1}{2}B(\frac{1}{2}, 2) = \frac{\Gamma(\frac{1}{2})\Gamma(2)}{2\Gamma(\frac{1}{2}+2)} = \frac{\sqrt{\pi}\times 1!}{2\times\frac{4!\sqrt{\pi}}{4^2\times 2!}} = \frac{2}{3}$

1951 Leaving Certificate Mathematics I Honours Q11ii

$\int_0^a \frac{x^{\frac{3}{2}}}{\sqrt{a-x}}\, dx = \int_0^a \frac{(ax)^{\frac{3}{2}}}{\sqrt{a-ax}}\, d(ax) = a^2 \int_0^1 x^{\frac{3}{2}}(1-x)^{-\frac{1}{2}}\, dx = a^2 B(\frac{5}{2}, \frac{1}{2}) = \frac{a^2\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{5}{2}+\frac{1}{2})} = \frac{a^2\times\frac{4!\sqrt{\pi}}{4^2\times 2!}\times\sqrt{\pi}}{2!} = \frac{3\pi a^2}{8}$

1956 Leaving Certificate Mathematics I Honours Q8ii

$\int_0^t x^2(t-x)^{\frac{3}{2}}\, dx = \int_0^t (tx)^2(t-tx)^{\frac{3}{2}}\, d(tx) = t^{\frac{9}{2}} \int_0^1 x^2(1-x)^{\frac{3}{2}}\, dx = t^{\frac{9}{2}} B(3, \frac{5}{2}) = \frac{t^{\frac{9}{2}}\Gamma(3)\Gamma(\frac{5}{2})}{\Gamma(3+\frac{5}{2})} = \frac{t^{\frac{9}{2}}\times 2!\times\frac{4!\sqrt{\pi}}{4^2\times 2!}}{\frac{10!\sqrt{\pi}}{4^5\times 5!}} = \frac{16t^{\frac{9}{2}}}{315}$

1962 Leaving Certificate Mathematics I Honours Q2ii

$\int_0^{\frac{\pi}{2}} \sin^6\theta\, d\theta = \frac{1}{2}B(\frac{7}{2}, \frac{1}{2}) = \frac{\Gamma(\frac{7}{2})\Gamma(\frac{1}{2})}{2\Gamma(\frac{7}{2}+\frac{1}{2})} = \frac{\frac{6!\sqrt{\pi}}{4^3\times 3!}\times\sqrt{\pi}}{2\times 3!} = \frac{5\pi}{32}$

1980 HSC 4 unit Q7iia

$\int_0^1 x(1-x)^{20}\, dx = B(2, 21) = \frac{\Gamma(2)\Gamma(21)}{\Gamma(2+21)} = \frac{1!\times 20!}{22!} = \frac{1}{462}$

1980 HSC 4 unit Q7iib

$\int_0^{\frac{\pi}{2}} \cos^2 x\, dx = \frac{1}{2}B(\frac{1}{2}, \frac{3}{2}) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{3})}{2\Gamma(\frac{1}{2}+\frac{3}{2})} = \frac{\sqrt{\pi}\times\frac{2!\sqrt{\pi}}{4^1\times 1!}}{2\times 1!} = \frac{\pi}{4}$

1985 HSC 4 unit Q1iii

$\int_0^1 x(1-x)^{99}\, dx = B(2, 100) = \frac{\Gamma(2)\Gamma(100)}{\Gamma(2+100)} = \frac{1!\times 99!}{101!} = \frac{1}{10100}$

1987 HSC 4 unit Q1iiib

$\int_0^{\frac{\pi}{2}} \cos^5 x\, dx = \frac{1}{2}B(\frac{1}{2}, 3) = \frac{\Gamma(\frac{1}{2})\Gamma(3)}{2\Gamma(\frac{1}{2}+3)} = \frac{\sqrt{\pi}\times 2!}{2\times\frac{6!\sqrt{\pi}}{4^3\times 3!}} = \frac{8}{15}$

1991 HSC 4 unit Q1ciii

$\int_0^{\frac{\pi}{2}} \sin^4 x\, dx = \frac{1}{2}B(\frac{5}{2}, \frac{1}{2}) = \frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{2\Gamma(\frac{5}{2}+\frac{1}{2})} = \frac{\frac{4!\sqrt{\pi}}{4^2\times 2!}\times\sqrt{\pi}}{2\times 2!} = \frac{3\pi}{16}$

1995 HSC 4 unit Q7a

$\int_0^{\frac{\pi}{2}} \sin^{2n} x\, dx = \frac{1}{2}B(\frac{2n+1}{2}, \frac{1}{2}) = \frac{\Gamma(\frac{2n+1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\frac{2n+1}{2}+\frac{1}{2})} = \frac{\frac{(2n)!\sqrt{\pi}}{4^n n!}\times\sqrt{\pi}}{2n!} = \frac{(2n)!\pi}{2^{2n+1}(n!)^2} = \frac{\pi\binom{2n}{n}}{2^{2n+1}}$

$= \frac{2n\times(2n-1)\times(2n-2)\times\cdots\times 1}{(2n\times(2n-2)\times\cdots\times 2)^2} \times \frac{\pi}{2}$ then even terms in numerator cancel

$= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$

and

$\int_0^{\frac{\pi}{2}} \sin^{2n+1} x\, dx = \frac{1}{2}B(n+1, \frac{1}{2}) = \frac{\Gamma(n+1)\Gamma(\frac{1}{2})}{2\Gamma(n+1+\frac{1}{2})} = \frac{n!\sqrt{\pi}}{\frac{2(2n+2)!\sqrt{\pi}}{4^{n+1}(n+1)!}} = \frac{2^{2n}(n!)^2}{(2n+1)!} = \frac{2^{2n}}{(2n+1)\binom{2n}{n}}$

$= \frac{((2n)\times(2n-2)\cdots\times 2)^2}{(2n+1)\times(2n)\cdots\times 1}$ then even terms in denominator cancel

$= \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdot \cdots \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1$

2020 HSC Extension 2 Q16biii

$\int_0^1 x^n(1-x)^n\, dx = B(n+1, n+1) = \frac{\Gamma(n+1)\Gamma(n+1)}{\Gamma(n+1+n+1)} = \frac{(n!)^2}{(2n+1)!}$

2023 HSC Extension 2 Q15aii

From 1995 Q7a, $\int_0^{\frac{\pi}{2}} \sin^{2n+1} x\, dx = \frac{1}{2}B(n+1, \frac{1}{2}) = \frac{2^{2n}(n!)^2}{(2n+1)!}$

and from 2020 Q16biii, $\int_0^1 x^n(1-x)^n\, dx = B(n+1, n+1) = \frac{(n!)^2}{(2n+1)!}$

Hence $\int_0^1 x^n(1-x)^n\, dx = \frac{1}{2^{2n}} \int_0^{\frac{\pi}{2}} \sin^{2n+1} x\, dx$