BAULKHAM HILLS HIGH SCHOOL

TRIAL 2013

YEAR 12 TASK 4

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 180 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- Show all necessary working in Questions 11-16
- Marks may be deducted for careless or badly arranged work

Total marks – 100
Exam consists of 11 pages.

This paper consists of TWO sections.

<u>Section 1</u> – Page 2-4 (10 marks)
Questions 1-10

- Attempt Question 1-10
- Allow about 15 minutes for this section

<u>Section II</u> – Pages 5-10 (90 marks)

- Attempt questions 11-16
- Allow about 2 hours 45 minutes for this section

Table of Standard Integrals is on page 11

Section I - 10 marks
Allow about 15 minutes for this section
Use the multiple choice answer sheet for question 1-10

1. Which of the following is equal to $\cos\theta$?

(A) $\dfrac{\sin\theta}{\tan\theta}$

(B) $\sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2}$

(C) $2\cos^2\theta - 1$

(D) $2\cos^2\frac{\theta}{2} + 1$

2. In Cartesian form $2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ is

(A) $-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

(B) $-i$

(C) $-\sqrt{2}(1 - i)$

(D) $\sqrt{2}(1 - i)$

3 Using an appropriate substitution

$\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2 x}{(1 + \tan x)^2}\, dx$ is equivalent to:

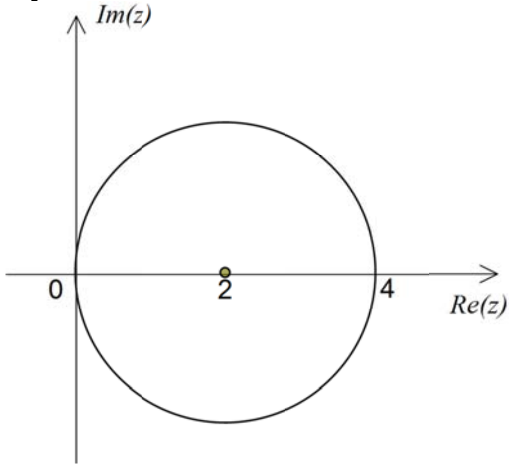
(A) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{u}\, du$

(B) $\displaystyle\int_0^2 \frac{u^2}{(1 + u)^3}\, du$

(C) $\displaystyle\int_0^2 \frac{1}{u^2}\, du$

(D) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{u^3}\, du$

4. Which of the following is the equation of the circle shown below?



(A) $(z+2)(\bar{z}+2)=4$

(B) $(z-2)(\bar{z}-2)=4$

(C) $(z+2i)(\bar{z}-2i)=4$

(D) $(z+2)(\bar{z}-2)=4$

5. Using implicit differentiation on the equation $y^3 = x^2 + xy$, then $\frac{dy}{dx}$ would equal

(A) $\frac{3y^2-2x}{x}$

(B) $\frac{2x+y}{3y^2-x}$

(C) $\frac{2x-y}{3y^2+y}$

(D) $\frac{2x}{3y^2+y}$

6. A satellite in a circular orbit around Earth, at a distance of 12000 km from Earth's centre makes 12 revolutions per day. Find the tangential speed of the satellite in km/h.

(A) π

(B) $\frac{72000}{\pi}$

(C) 12000π

(D) $12000\pi^2$

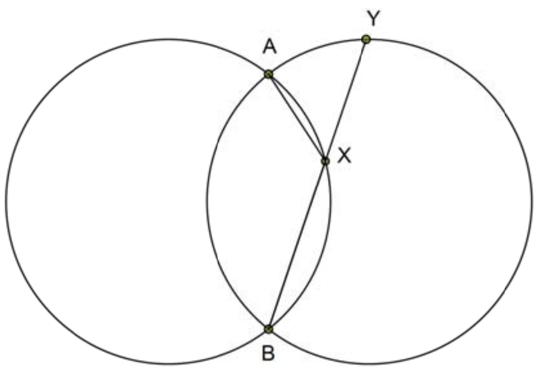
7.	If α, β and γ are the roots of the equation $x^3 - 3x + 4 = 0$ Then the cubic with roots α^2, β^2 and γ^2 *is* (A) $8x^3 - 9x + 4 = 0$ (B) $x^3 - 6x^2 + 9x - 16 = 0$ (C) $x^3 + 9x^2 - 12x + 4 = 0$ (D) $8x^3 + 4x^2 - 9x + 16 = 0$
8.	Given $(2i + 1)$ is a root of the equation $x^3 - 4x^2 + 9x - 10 = 0$ then another root is (A) 2 (B) 5 (C) $2i - 1$ (D) 10
9.	$\tan(\cos^{-1} x)$ is equal to (A) $-\frac{\sqrt{1-x^2}}{x}$ (B) $-\frac{x}{\sqrt{1-x^2}}$ (C) $\frac{\sqrt{1-x^2}}{x}$ (D) $\frac{x}{\sqrt{1-x^2}}$
10.	$\int x\sqrt{1-x}\,dx$ equals (A) $-\frac{1}{3}x^2(1-x)^{\frac{3}{2}} + c$ (B) $\frac{1}{3}x^2(1-x)^{\frac{3}{2}} + c$ (C) $-\frac{2}{5}x(1-x)^{\frac{5}{2}} + \frac{2}{3}(1-x)^{\frac{3}{2}} + c$ (D) $\frac{2}{5}(1-x)^{\frac{5}{2}} - \frac{2}{3}(1-x)^{\frac{3}{2}} + c$
	End of Section 1

Section II – Extended Response

Attempt questions 11-16.
Answer each question on a SEPARATE PAGE. Clearly indicate question number.
Each piece of paper must show your name.
All necessary working should be shown in every question.

Question 11 (15 marks)		**Marks**
a)	Let $z_1 = 3 - 4i$ and $z_2 = -3 + 2i$	
	(i) $z_1 - \overline{z_2}$	1
	(ii) $\frac{z_1}{z_2}$	2
b)	Given that $(1-2i)^2 = -3-4i$, solve $z^2 - 5z + (7+i) = 0$.	2
c)	On an Argand diagram, shade the region specified by the conditions $\lvert z - 6 + 5i \rvert \leq 3$ and $\text{Re}(z) \leq 6$.	2
d)	If $z = a(\cos\theta + i\sin\theta)$ when a and θ are real, show that $\frac{z}{z^2+a^2}$ is equivalent to $\frac{1}{2a\cos\theta}$	3
e)	(i) Prove that if $y = \left(x + \sqrt{1+x^2}\right)^m$ then $\frac{1}{y} \cdot \frac{dy}{dx} = \frac{m}{\sqrt{1+x^2}}$	2
	(ii) Show $\frac{d^2y}{dx^2} = \frac{m^2y\sqrt{1+x^2} - myx}{(1+x^2)\sqrt{1+x^2}}$	2
	(iii) Prove that $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2y = 0$	1
	End of Question 11	

Question 12 (15 marks)		**Marks**
a)	Find $\displaystyle\int \frac{dx}{(x+1)(x^2+2)}$	**3**
b)	(i) Show that $\log_{ab} x = \dfrac{\log_a x}{1+\log_a b}$	**2**
	(ii) Hence show that $\log_2 5 = \dfrac{1-\log_{10} 2}{\log_{10} 2}$	**1**
c)	Consider the curves $$\frac{x^2}{16}+\frac{y^2}{7}=1 \quad \text{and} \quad x^2-\frac{y^2}{8}=1$$	
	(i) Show that both curves have the same focii.	**3**
	(ii) Find the equation of the circle that passes through the points of intersection of these two curves.	**3**
d)	(i) In how many distinct ways can the letters of the word **A N G L E** be arranged.	**1**
	(ii) If these arrangements are listed in alphabetical order, in which place (ie. 1st, 2nd, 3rd, etc…) is the word ANGLE .	**2**
	End of Question 12	

Question 13 (15 marks)		**Marks**
a)	$$I_n = \int_0^{\pi} \sin^n x \, dx$$	
	(i) Prove that $I_n = \frac{n-1}{n} I_{n-2}$	3
	(ii) Hence evaluate I_5	2
b)	When a polynomial $P(x)$ is divided by $(x-3)$ and $(x-7)$ the respective remainders are 3 and 5. Find the remainder when $P(x)$ is divided by $(x-3)(x-7)$.	3
c)	Two circles of equal radii intersect at A and B. X is a point on the circle between A and B and BX is produced to meet the second circle at Y.  Copy the diagram in your booklet and prove that $AX = AY$, showing any necessary constructions.	3
d)	Find the volume of the solid of revolution generated when the area enclosed between the curve $y = 4 - x^2$ and the lines $y = 4$ and $x = 2$ is rotated about the line $x = 2$.	4
	End of Question 13	

Question 14 (15 marks)	**Marks**

a) A body of unit mass falls under gravity through a resistive medium.
The body falls from rest from a cliff 50 metres above the ground.

The resistance to its motion is $\frac{v^2}{100}$ where v m s^{-1} is the speed of the body when it has fallen a distance of x metres.

(i) Show that the equation of the motion is $\ddot{x} = g - \frac{v^2}{100}$ **1**

(ii) Show that the terminal velocity V of the body is given by **1**

$$V = 10\sqrt{g}\ \text{ms}^{-1}$$

(iii) Show that $v^2 = V^2\left(1 - e^{-\frac{x}{50}}\right)$. **3**

(iv) How far has the body fallen when it reaches a velocity of $\frac{V}{2}$. **2**

(v) Find the velocity reached in terms of the terminal velocity when the body hits the ground. **2**

(vi) If $v = v_1$ when $x = d$ and $v = v_2$ when $x = 2d$, show that **2**

$$v_2^2 = v_1^2\left(2 - \frac{v_1^2}{V^2}\right)$$

b) The equation $x^4 - 5x^3 - 9x^2 + ax + b = 0$ has a triple root.

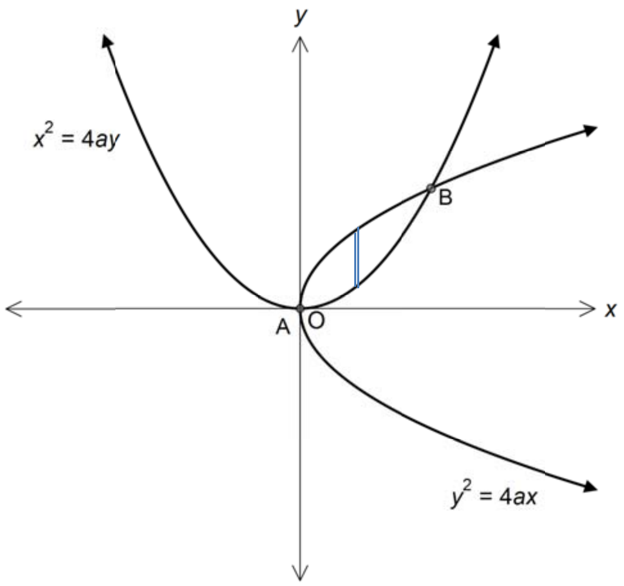
Given that this root is an integer:

(i) find the triple root. **2**

(ii) find the value of b. **2**

End of Question 14

Question 15 (15 marks)			Marks
a)	(i)	Prove by mathematical induction that $(1+x)^n - 1$ is divisible by x for all integers $n \geq 1$.	3
	(ii)	By factorising $35^n - 7^n - 5^n + 1$ and using part (i), prove that $35^n - 7^n - 5^n + 1$ is divisible by 24.	2
b)		Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{5+4\cos x}$	4
c)		The base of a certain solid is the region bounded by the curve $y^2 = 4ax$ and $x^2 = 4ay$ and cross sections to the plane perpendicular to the x-axis are semi circles.	



	(i)	Show that the two curves intersect at $A(0,0)$ and $B(4a, 4a)$.	1
	(ii)	Show that the cross sectional area, A, of a typical slice is $A = \frac{\pi}{2}(\sqrt{ax} - \frac{x^2}{8a})^2$.	2
	(iii)	Hence find the volume of the solid formed.	3

End of Question 15

Question 16 (15 marks)		
a)	P is a point $\left(p, \frac{1}{p}\right)$ on the rectangular hyperbola $xy = 1$. The line PO is produced to point Q also on the rectangular hyperbola. A circle centre P and radius PQ is drawn to cut the hyperbola at A, B, C and Q. (i) Prove that the parameters of the points of intersection of the circle and the hyperbola are given by the equation $$p^2t^4 - 2p^3t^3 - 3(p^4+1)t^2 - 2pt + p^2 = 0$$ (ii) Deduce that $t_A + t_B + t_C = 3p$ where t_A, t_B and t_C are the parameters at A, B and C	3 2
b)	(i) Show that $$(1 + i\tan\theta)^n + (1 - i\tan\theta)^n = \frac{2\cos n\theta}{\cos^n\theta}$$ where $\cos\theta \neq 0$ and n is a positive integer. (ii) Hence show that if z is a purely imaginary number, the roots of $(1+z)^4 + (1-z)^4 = 0$ are $z = \pm i\tan\frac{\pi}{8}, \pm i\tan\frac{3\pi}{8}$.	2 3
c)	Consider the sequence defined by $$V_k = \frac{1}{2k+1} + \frac{1}{2k+2} + \cdots + \frac{1}{3k}$$ where k is a positive integer (i) Show that $V_k < \frac{1}{2}$ (ii) Given that $p < x < p+1$, where x is a real number and p is a positive integer show that $\frac{1}{p+1} < \int_p^{p+1} \frac{dx}{x} < \frac{1}{p}$ (iii) Hence show that $$\int_{2k+1}^{3k+1} \frac{dx}{x} < V_k < \int_{2k}^{3k} \frac{dx}{x}$$ (iv) Hence find the limit of V_k as $k \to \infty$	 1 1 2 1
	End of Paper	

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \ n \neq -1; \ x \neq 0, \ if \ n < 0$$

$$\int \frac{1}{x} dx = \ln x, \qquad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \ a \neq 0$$

$$\int \cos ax dx = \frac{1}{a}\sin ax, \ a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a}\cos ax, \ a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a}\tan ax, \ a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a}\sec ax, \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \ a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}), \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2})$$

NOTE: $\ln x = \log_e x, \ x > 0$