BAULKHAM HILLS HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2010

MATHEMATICS

EXTENSION 2

GENERAL INSTRUCTIONS:

- Reading time − 5 minutes.
- Working time − 3 hours.
- Write using blue or black pen.
- Start each question on a new page.
- Write your Student Number at the top of each page.
- Calculators may be used.
- A table of standard integrals is provided.
- **ALL** necessary working should be shown in every question.

QUESTION 1 (15 marks)

Marks

(a) Find each of the following integrals

(i) $\int x^2(1+2x^3)^{-5}dx$ 2

(ii) $\int tan^4 x dx$ 3

(iii) $\int \frac{dx}{3+2\cos x}$ 4

(b) Find $\int \frac{e^x+e^{2x}}{1+e^{2x}}dx$ 3

(c) (i) Show that $\sin(A+B)+\sin(A-B)=2sinAcosB$ 1

(ii) Hence calculate $\int sin5xcos4xdx$ 2

QUESTION 2 (15 marks)

(a) For $z_1=2-3i$ and $z_2=1+5i$ find, in the form *a*+*ib*, the values of

(i) $z_1+\overline{z}_2$ 2

(ii) $z_1 z_2$ 2

(iii) $\frac{z_1}{z_2}$ 2

(b) (i) Solve $(x+iy)^2=6i$ 2

(ii) Hence or otherwise solve $z^2-(1-i)z-2i=0$ 3

(c) (i) Express $z=1-\sqrt{3}i$ in modulus-argument form 2

(ii) Hence express z^6 in the form $a+ib$ 2

QUESTION 3 (15 marks)

Marks

(a) The hyperbola H has the equation $\frac{x^2}{4} - \frac{y^2}{12} = 1$

Find (i) its eccentricity 2

(ii) the coordinates of its foci 1

(iii) the equations of its directrices 1

(iv) the equations of its asymptotes 1

(b) (i) Show that the gradient of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point P$(a\cos\theta, b\sin\theta)$ is $\frac{-b\cos\theta}{a\sin\theta}$ 2

(ii) Hence show that the equation of the tangent is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ 3

(iii) Show that the x-intercept of this tangent is $(\frac{a}{\cos\theta}, 0)$ 1

(iv) Hence, or otherwise, find the points on the curve $4x^2 + 3y^2 = 12$ whose tangent passes through (2,0) 4

QUESTION 4 (15 marks)

(a) OABC is a square on the Argand diagram and is labeled in an anticlockwise direction. A represents $z = a + ib$ and B represents $4 + 7i$.

(i) Find, in terms of a and b, the complex number represented by C. 2

(ii) Hence evaluate a and b. 2

(b) The equation $x^3 + 2x - 1 = 0$ has roots α, β and γ. Find the equation with roots:

(i) $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$ 2

(ii) α^2, β^2 and γ^2 2

(c) (i) Show that 2 is a root of multiplicity 3 for $P(x) = x^4 - 3x^3 - 6x^2 + 28x - 24$ 2

(ii) Hence solve $P(x) = 0$ 2

(d) Draw on separate argand diagrams the following loci:

(i) $z\bar{z} = 3$ 1

(ii) $\arg\left(\frac{z}{z-1}\right) = \frac{\pi}{3}$ 2

QUESTION 5 (15 marks)

Marks

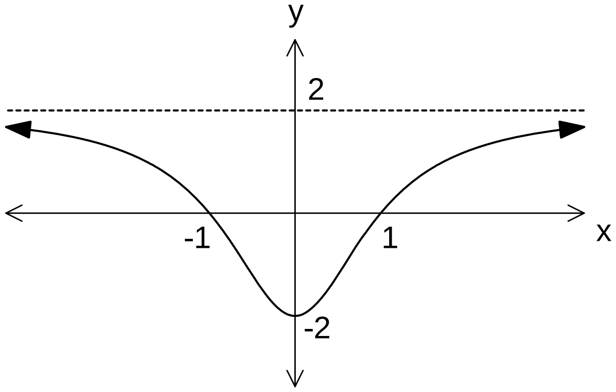
(a) Suppose $x>0, y>0, z>0$

(i) Prove $x^2 + y^2 + z^2 \geq xy + yz + xz$ 2

(ii) Given $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - xz)$
prove $x^3 + y^3 + z^3 \geq 3xyz$ 1

(iii) Hence show $a + b + c \geq 3(abc)^{\frac{1}{3}}$ 1

(b) Given below is the graph of $f(x) = 2 - \frac{4}{x^2+1}$.



Use the graph of $y = f(x)$ to sketch, on separate axes, the graphs of

(i) $y = |f(x)|$ 2

(ii) $y = [f(x)]^2$ 2

(iii) $y^2 = f(x)$ 2

(iv) $y = \frac{1}{f(x)}$ 2

(c) For the curve $x^3 + 3x^2y - 2y^3 = 16$

(i) Show that $\frac{dy}{dx} = \frac{x^2+2xy}{2y^2-x^2}$ 1

(ii) Find the coordinates of the stationary points on the curve 2

QUESTION 6 (15 marks)

(a) Find, using slices, the volume generated when the area bounded by $y = x^2$ and the line $y = 3$ is rotated about the line $y = 3$. 4

(b) Find, using cylindrical shells, the volume obtained by revolving about the y-axis the region bounded by the curve $y = sinx$, for $0 \leq x \leq \pi$, and the x-axis. 4

(c) A solid has a semi-circular base whose equation is $= \sqrt{4 - x^2}$. Vertical cross-sections, perpendicular to the diameter, are right-angled triangles whose height is bounded by the parabola $z = 4 - x^2$.

(i) Draw a neat diagram, including a typical slice, representing this information. 1

(ii) By slicing at right angles to the x-axis, show that the volume of the solid is given by $\int_0^2 (4 - x^2)^{\frac{3}{2}} dx$. 3

(iii) Hence calculate this volume. 3

QUESTION 7 (15 marks)

Marks

(a) Use the method of partial fractions to show that $\int_0^1 \frac{6x+4}{(x^2+1)(x+1)} dx = \frac{5\pi}{4} - \frac{1}{2}\log_e 2$ 4

(b) Let $P(z) = z^4 + bz^2 + d$ where b and d are real numbers and $d \neq 0$. *P(z)*has a double zero α.

(i) Prove $P'(z)$ is odd. 2

(ii) Prove that $-\alpha$ is also a double zero of *P(z)*. 2

(c) A mass of 35 kg is dropped from a balloon falling at 30 m/s. The mass experiences air resistance measuring 70*v* Newtons, where *v* m/s is its velocity. Take *g* as 10m/s^2.

(i) Show that the velocity of the mass *t* seconds after being dropped, but before hitting the ground, is given by $v = 5 + 25e^{-2t}$. 3

(ii) Describe what happens to the velocity as $t \to \infty$. 1

(iii) If the mass was dropped from 400m above the ground, how close to the ground will it be after 1 minute? 3

QUESTION 8 (15 marks)

(a) Given $I_n = \int_0^1 x^n e^{-x} dx$

(i) Calculate I_0 2

(ii) Prove $I_n = nI_{n-1} - \frac{1}{e}$ 2

(iii) Hence find $\int_0^1 x^3 e^{-x} dx$ 2

(b) Two particles of equal mass are attached to the ends A and B of a light inextensible string which passes through a small hole at the apex C of a hollow right circular cone fixed with its axis vertical and apex on top. The semi-vertical angle of the cone is θ. The particle at A, where AC is a units, moves in a horizontal circle with constant angular velocity ω on the smooth surface of the cone, while the other particle at B hangs at rest inside the cone.

(i) Represent this information on a diagram showing relevant forces. 1

(ii) Show that $\omega^2 = \frac{g}{a(1+\cos\theta)}$ 2

(iii) Hence, or otherwise, deduce that $\frac{g}{2\omega^2} < a < \frac{g}{\omega^2}$ 2

(c) If $x>0$, prove $x - \frac{1}{3}x^3 < \tan^{-1} x < x - \frac{1}{3}x^3 + \frac{1}{3}x^5$ 4

END OF EXAM