STUDENT'S NAME: ______________________________

TEACHER'S NAME: ______________________________

BAULKHAM HILLS HIGH SCHOOL

.

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2007

MATHEMATICS

EXTENSION 2

GENERAL INSTRUCTIONS:

- Reading time − 5 minutes.
- Working time − 3 hours.
- Write using blue or black pen.
- Start each question on a new page.
- Write your Student Number at the top of each page.
- Calculators may be used.
- A table of standard integrals is provided.
- **ALL** necessary working should be shown in every Question.

QUESTION 1 (15 marks)

Marks

(a) Simplify $\sin(A-B)+\sin(A+B)$ and hence find

$$\int \sin 5x \cos 3x \, dx$$ 3

(b) Find real constants A, B, C such that:

$$\frac{x^2+5x+2}{(x^2+1)(x+1)} \equiv \frac{\mathrm{Ax}+\mathrm{B}}{x^2+1} + \frac{C}{x+1}$$

and hence find $\int \frac{x^2+5x+2}{\left(x^2+1\right)\left(x+1\right)} dx$ 3

(c) Find $\int \sin^{-1} x \, dx$ 3

(d) Find $\int \sqrt{\frac{1+x}{1-x}} dx$ 3

(e) Evaluate $\int_0^1 \mathrm{x}^5 e^{x^3} \mathrm{dx}$ 3

QUESTION 2 (15 marks)

(a) If $z = 2\sqrt{3}\, i - 2$ find:

(i) $|z|$ 1

(ii) $\arg z$ 1

(iii) Re $(1+2i)\, \overline{z}$ 2

b) If ω is a complex root of the equation

$z^3 = 1$:

(i) Show that $1+\omega+\omega^2 = 0$ 2

(ii) Find the value of $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)$ 3

(c) Find the equation of the locus of z where $|z-2i| = \text{Im}\, z$. 3

(d) Given $z = 2 - i$, find real values of a and b such that: 3

$$a\,z + \frac{b}{z} = 1$$

QUESTION 3 (15 marks)

Marks

(a) Sketch the curve:

$f(x)=\frac{(x+1)(x+3)}{x}$ showing the stationary points and asymptotes. 3

Hence draw neat half page sketches of:

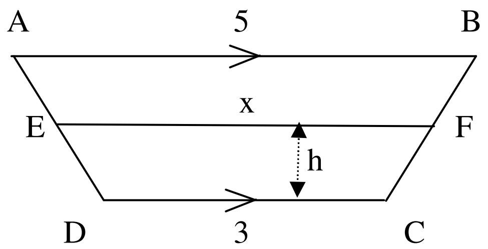
(i) $y=\frac{1}{f(x)}$ 2

(ii) $y=|f(x)|$ 1

(iii) $y=\sqrt{f(x)}$ 1

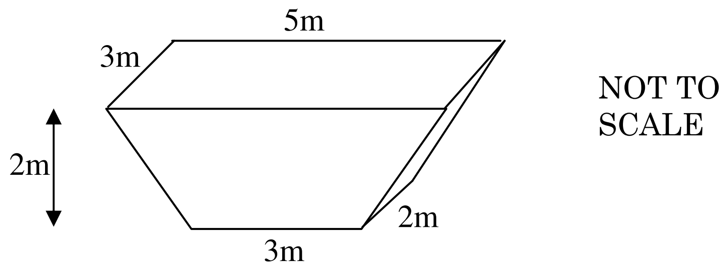
(iv) $y=\log_e f(x)$ 2

(b)



A trapezium ABCD has parallel sides AB = 5 m and CD = 3m, 2m apart. E lies on AD and F lies on BC such that EF is parallel to DC. The distance from EF to DC is h m and EF = x show that:

$x = 3 + h$. 1



The diagram is of a waste bin with a rectangular base of side 3m and 2m. Its top is also rectangular, parallel to the base with dimensions 5m and 3m. The bin has a depth of 2m, each of its four sides are trapeziums. Find the volume of the bin. 5

QUESTION 4 (15 marks)

Marks

(a) (i) Prove that the normal at point $P\left(cp, \frac{c}{p}\right)$ on the curve $xy = c^2$

is $p^3x - py = c\left(p^4 - 1\right)$ 2

(ii) The normal at P meets the hyperbola again at point $Q\left(cq, \frac{c}{q}\right)$. Prove that 2

$p^3q = -1$

(iii) The tangent at P meets the y axis at R. 3

Show that the area of the triangle PQR is:

$$A = \frac{c^2}{2}\left(p^2 + \frac{1}{p^2}\right)^2$$

and hence find the minimum area of this triangle. 2

(b) (i) Evaluate $\int_0^{\frac{\pi}{2}} (\sin t)^{2k} \cos t \, dt$ 1

(ii) Noting that $(\cos t)^{2n+1} = \cos t\left(1 - \sin^2 t\right)^n$ and by using the Binomial Theorem to expand $\left(1 - \sin^2 t\right)^n$ where n is a positive integer, show that

$$\int_0^{\frac{\pi}{2}} (\cos t)^{2n+1} dt = \sum_{r=0}^{n} (-1)^r \frac{1}{2r+1} \cdot {}^nC_r$$ 3

(iii) Use the result of part (ii) to evaluate

$$\int_0^{\frac{\pi}{2}} \cos^7 t \, dt$$ 2

QUESTION 5 (15 marks)

Marks

(a) The quadratic equation $x^2 - x + k = 0$ where k is a real number has 2 distinct positive roots α *and* β.

Show that:

(i) $0 < k < \frac{1}{4}$ 2

(ii) $\alpha^2 + \beta^2 = 1 - 2k$ and hence deduce that $\alpha^2 + \beta^2 > \frac{1}{2}$ 3

(iii) $\frac{1}{\alpha^2} + \frac{1}{\beta^2} > 8$ 2

(b) Let α, β and γ be the roots of:

$x^3 - 7x^2 + 18x - 7 = 0$

Find the polynomial with roots:

(i) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ 2

(ii) $(1+\alpha^2), (1+\beta^2), (1+\gamma^2)$ 3

(c) By considering the stationary values of:

$f(x) = x^3 - 3px^2 + 4q$, where p and q are positive real constants, show

that the equation $f(x) = 0$ has 3 real distinct roots if

$p^3 > q$. 3

QUESTION 6 (15 marks)

Marks

(a) A particle of mass m is moving vertically in a resisting medium in which the resistance to the motion has a magnitude of $\frac{1}{10}$ m v^2 where the particle has speed u ms^{-1}. The acceleration due to gravity is g ms^{-2}.

(i) If the particle falls vertically downwards from rest, show that its acceleration is given by:

$$a = g - \frac{1}{10}v^{2.}$$

Hence show that its terminal speed V ms^{-1} is given by

$$V = \sqrt{10g}\ .$$ 2

(ii) If the particle is projected vertically upwards with speed $V\tan\alpha\, ms^{-1}$ $(0 < \alpha < \frac{\pi}{2})$ show that its acceleration $a\,ms^{-2}$ is given by

$$a = -\left(g + \frac{v^2}{10}\right)$$

Hence show that it reaches a maximum height H metres given by :

$$H = 5\log_e \sec^2\alpha$$ 4

and that it returns to its point of projection with speed

$$V\sin\alpha\, ms^{-1}.$$ 4

(b) The roots of the equation $4x^3 - 36x^2 + 107x + k = 0$ are in arithmetic progression, find:

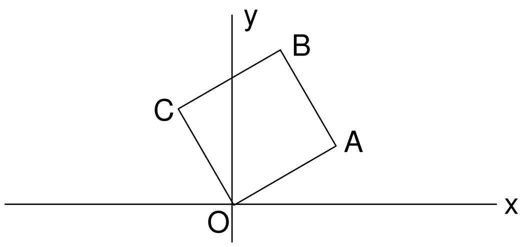
(i) k 2

(ii) the roots of the equation. 3

QUESTION 7 (15 marks)

Marks

(a) In the square OABC shown below, the point A represents 4 + 3i. What complex numbers do the points B and C represent. 3



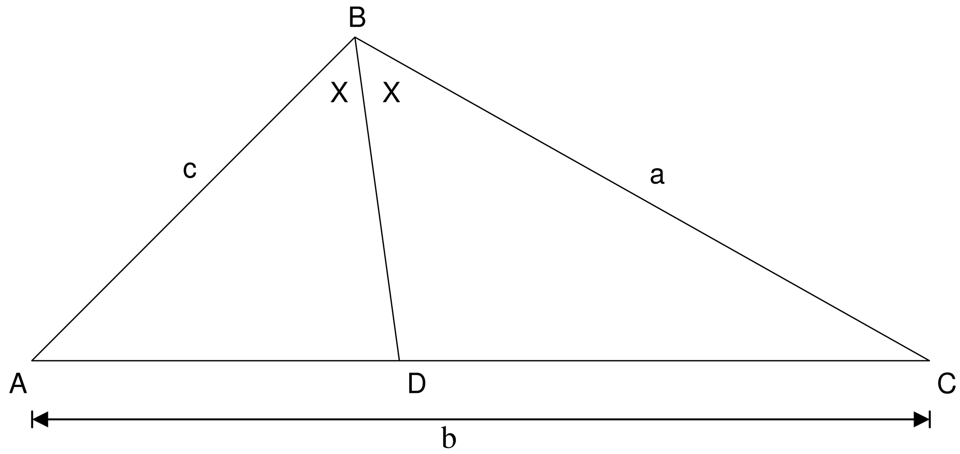
(b) (i) Determine the real values of k for which the equation:

$\frac{x^2}{19 - k} + \frac{y^2}{7 - k} = 1$ defines an ellipse and a hyperbola respectively. 2

(ii) Sketch the curve corresponding to the value of k = 3, showing foci, directrices and where the curve cuts the coordinate axes. 4

(iii) Describe how the shape of this curve changes as k varies from 3 to 7. 1

(c)



In ΔABC, BD bisects $\angle ABC$ as shown in the diagram.

(i) By considering the area of Δ ABC, show that 2

$$BD = \frac{2ac \cos x}{a+c}$$

(ii) Show that : $\cos x = \frac{1}{2}\sqrt{\frac{(a+c)^2 - b^2}{ac}}$ 2

(iii) Hence show that $BD = \frac{\sqrt{ac}}{a+c}\sqrt{(a+c)^2 - b^2}$ 1

QUESTION 8 (15 marks)

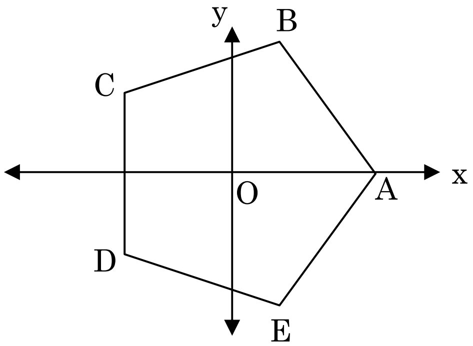
Marks

(a) If $I_n = \int_1^e x^3 (\log_e x)^n \, dx \quad for\ n = 0, 1, 2, ...$ show that

$I_n = \frac{e^4}{4} - \frac{n}{4} I_{n-1}$ and hence find the value of

$\int_1^e x^3 (\log_e x)^2 \, dx$ 4

(b)



In the diagram, the complex numbers z_0, z_1, z_2, z_3 *and* z_4 are represented by the vertices of a regular pentagon with center O and vertices A,B,C,D and E respectively. Given that $z_0 = 2$

(i) Express z_2 in modulus argument form 1

(ii) Find the value of $(z_2)^5$ 2

(iii) Show that the perimeter of the pentagon is $20 \sin \frac{\pi}{5}$ 2

(c) (i) Use De Moivre's Theorem to find expressions for $\cos 5\theta$ *and* $\sin 5\theta$ and hence show that: 3

$$\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$$

(ii) By considering the equation: 3

$$x^5 - 5x^4 - 10x^3 + 10x^2 + 5x - 1 = 0$$

Prove that $\tan\frac{\pi}{20} + \tan\frac{9\pi}{20} + \tan\frac{17\pi}{20} + \tan\frac{33\pi}{20} = 4$