BAULKHAM HILLS HIGH SCHOOL

2011
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions
- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Marks may be deducted for careless or badly arranged work

Total marks – 120
- Attempt Questions 1 – 8
- All questions are of equal value
- Start a separate piece of paper for each question.
- Put your student number and the question number at the top of each sheet.

Question 1 (15 marks) - Start on a new page

a) Find the following indefinite integrals

(i) $\int \cos^3 x\, dx$ **2**

(ii) $\int \frac{x-2}{x^2+1}\, dx$ **2**

(iii) $\int x \sin 2x \;\; dx$ **2**

b) Evaluate $\int_0^1 \frac{dx}{\sqrt{3-2x-x^2}}$ **3**

c) Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x+\cos x}$ **3**

d) Find $\int \frac{2x\, dx}{x^3-2x^2+9x-18}$ **3**

Question 2 (15 marks) - Start on a new page

a) (i) Express $z_1 = \dfrac{7+4i}{3-2i}$ in the form $a+ib$ where a, b are real **2**

(ii) On an Argand diagram sketch the locus of the point representing the complex number z such that $|z - z_1| = \sqrt{5}$ **1**

(iii) Prove that the locus passes through the origin and find the greatest value of $|z|$ **2**

b) Let $z = 2 + 3i$ and $w = 1 + i$ **2**
Find zw and $\frac{1}{w}$ in the form $x + iy$

c) (i) Express $(1 - \sqrt{3}i)$ in modulus argument form **2**

(ii) Hence write $\left(1 - \sqrt{3}i\right)^{10}$ in the form $x + iy$ **2**

d) The complex number $z = x + iy$ when x and y are real, is such that $|z - i| = \text{Im}(z)$

(i) Show that the locus of point P representing z has Cartesian equation $y = \frac{1}{2}(x^2 + 1)$ and sketch the locus **2**

(ii) By finding the gradients of the tangents to this curve which pass through the origin, find the set of possible values of $\arg z$ $(-\pi < \arg z \leq \pi)$ **2**

Question 3 (15 marks) - Start a new page **Mar[ks]**

a) Sketch the graph **2**

$y = (2x+1)(x+1)$ clearly showing all intercepts on the co-ordinate axes and the co-ordinates of any turning points.

b) Use the graph of part (a) to sketch the graphs below, showing clearly the intercepts on the co-ordinate axes, the co-ordinates of any turning points and the equation of any asymptotes.

(i) $y = \log_e[(2x+1)(x+1)]$ **2**

(ii) $y = \dfrac{1}{(2x+1)(x+1)}$ **2**

c) The region bounded by the curve

$$y = \frac{1}{(2x+1)(x+1)}$$

the co-ordinate axes and the line $x = 4$ is rotated through one complete revolution about the y axis.

(i) Use the method of cylindrical shells to express the volume of the resulting solid of revolution as an integral . **2**

(ii) Evaluate the integral in part (i). **4**

d) When $P(x) = x^4 + ax^3 + b$ is divided by $x^2 + 4$, the remainder is $-x + 13$. Find the values of a and b. **3**

Question 4 **(15 marks) - Start a new page**

a) A hyperbola has Cartesian equation

$$3x^2 - y^2 = 12$$

find (i) its eccentricity **1**

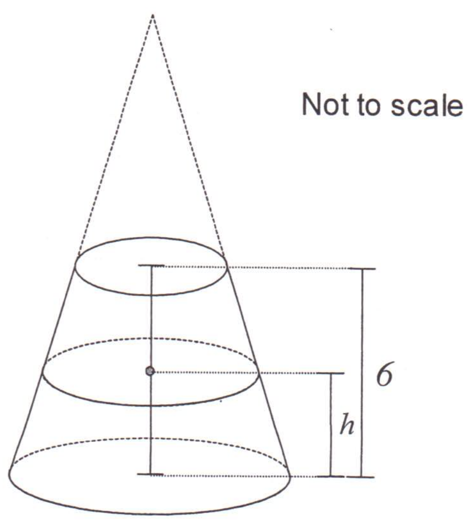
(ii) the co-ordinates of its foci **1**

(iii) the equations of the directrices **1**

(iv) the equations of the asymptotes **1**

hence sketch the hyperbola indicating all the features of your diagram. **1**

b) A right elliptical cone has its top cut off through a plane parallel to its elliptical base. The remaining solid has an ellipse as its base



The remaining solid has the ellipse $\frac{x^2}{9} + \frac{4y^2}{9} = 1$ as its base, and another ellipse $x^2 + 4y^2 = 1$ as its top.

The height of the solid is 6 units.

(i) Given that the area of an ellipse with equation $\frac{x^2}{9} + \frac{4y^2}{9} = 1$ is πab, show that the area of the ellipse at height h units above the base is **3**

$$A = \frac{\pi(h-9)^2}{18}$$

(ii) Hence find the volume of the soild. **3**

c) A plane curve is defined implicitly by $x^2 + 2xy + y^5 = 4$
This curve has a horizontal tangent at $P(x, y)$ show that $x = \alpha$ is a root of the equation $x^5 + x^2 + 4 = 0$ **4**

Question 5 (15 marks) - Start a new page | **Marks**

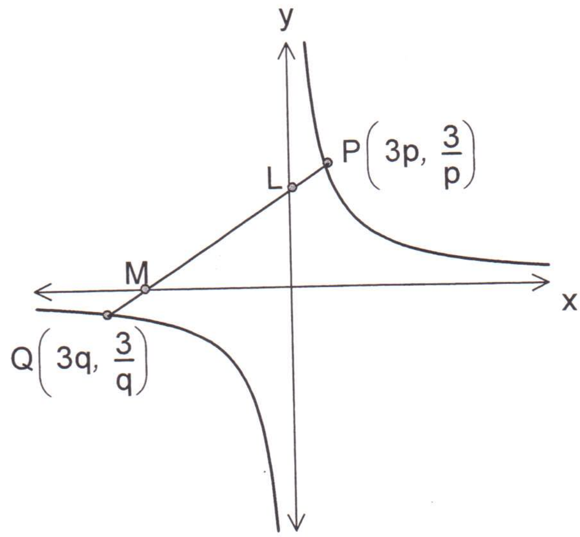
a) The equation $x^3 + px - 1 = 0$ has 3 non zero roots α, β, γ

(i) Find the values of $\alpha^2 + \beta^2 + \gamma^2$ and $\alpha^4 + \beta^4 + \gamma^4$ in terms of p and show that p must be negative. **4**

(ii) Find the monic equation with coefficients in terms of p where roots are **2**

$$\frac{\alpha}{\beta\gamma}, \frac{\beta}{\alpha\gamma}, \frac{\gamma}{\alpha\beta}$$

b) A chord PQ of the rectangular hyperbola $xy = 9$ meets the asymptotes at L and M as shown



i) Show that the equation of the chord PQ is $pqy + x = 3(p + q)$ **1**

ii) Find the co-ordinates of N the midpoint of PQ **2**

iii) Show that $PL = MQ$ **2**

iv) If the chord PQ is a tangent to the parabola $y^2 = 3x$ find the locus of N **2**

c) Solve in terms of a **2**

$a^x = e^{2x-1}$ where $a > 0, a \neq \frac{1}{e^2}$ e^2

Question 6 (15 marks) - Start a new page

a) Solve the equation **3**

$$x^4 - 6x^3 + 9x^2 + 6x - 20 = 0$$

given $(2 + i)$ is one of it's zeroes.

b) $P(a\cos\theta, b\sin\theta)$ lies on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad where\ a > b > 0.$$

The tangent and normal at point P cut y-axis at A and B respectively, and S is a focus of the ellipse

i) Show that $\angle ASB = 90°$ **2**

ii) Hence show that A, P, S and B are concyclic and state the coordinates of the centre of the circle through A, P, S and B. **3**

c) Prove that $\int_0^a f(x)\ dx = \int_0^a f(a - x)\ dx$ **2**

Hence evaluate $\int_0^\pi \frac{x\sin x}{1 + \cos^2 x} dx$ **3**

d) Draw a neat sketch of $y = \frac{1}{\sin^{-1} x}$ **2**

Question 7 (15 marks) - Start a new page **Mar**

a) Given that 1, ω and ω^2 are the three cube roots of unity

i) Find the value of $(1 + 2\omega + 3\omega^2)(1 + 2\omega^2 + 3\omega)$ **3**

ii) If the equations $x^3 - 1 = 0$ and $px^5 + qx + r = 0$ have a common root, evaluate **3**

$$(p + q + r)(p\omega^5 + q\omega + r)(p\omega^{10} + q\omega^2 + r)$$

b) i) Show that

$$\left(1 - \sqrt{x}\right)^{n-1}.\sqrt{x} = \left(1 - \sqrt{x}\right)^{n-1} - \left(1 - \sqrt{x}\right)^n$$ **1**

ii) If $I_n = \int_0^1 \left(1 - \sqrt{x}\right)^n dx$ for $n \geq 0$ **3**

Show that $I_n = \dfrac{n}{n+2} I_{n-1}$ $for\ n \geq 1$

And hence evaluate I_{100}

c) Prove that the volume, V, the area of the curved surface, S, and the radius of the base, r, of a right circular cone are connected by the equation **2**

$$9V^2 = r^2(S^2 - \pi^2 r^4)$$

3

Show that the maximum volume for a given curved surface area S, is

$$\frac{2^{\frac{1}{2}} S^{\frac{3}{2}}}{\pi^{\frac{1}{2}} 3^{\frac{7}{4}}}$$

Question 8 (15 marks) - Start a new page

a) Prove by mathematical induction that $7^n + 3n(7^n) - 1$ is divisible by 9. 3

b) i) Write the general solution of 1

$$\tan 4\theta = 1$$

ii) Use De Moivre's Thereom to find $\cos 4\theta$ and $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$ and hence determine the result 4

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

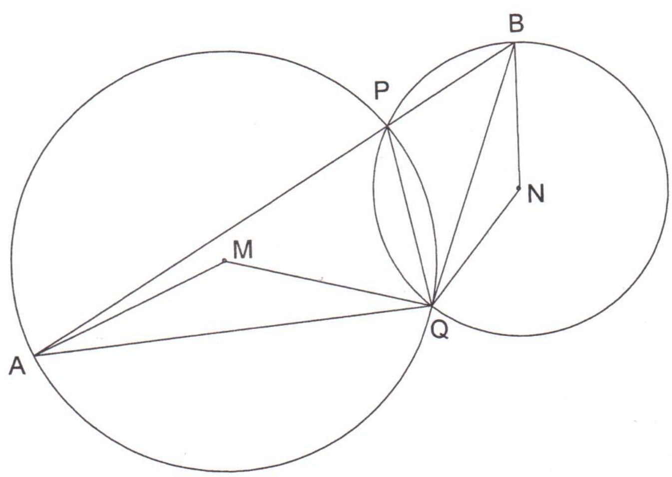
iii) Find the roots of 3

$x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

in the form $x = \tan\theta$ and hence prove that

$$\tan^2\frac{\pi}{16} + \tan^2\frac{3\pi}{16} + \tan^2\frac{5\pi}{16} + \tan^2\frac{7\pi}{16} = 28$$

c) 4



In the given diagram, two circles whose centres are M and N intersect in P and Q. A line drawn through P meets the two circles in A and B. Prove that $\angle MAQ = \angle NBQ$.

End of Exam