

Student Number _____

ASCHAM SCHOOL

2011
YEAR 12

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

Mathematics

Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.

Total marks – 120

- Attempt Questions 1-8.
- All questions are of equal value.
- Answer each question in a new booklet.

Total marks – 120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

QUESTION 1 (15 marks) **Marks**
Use a SEPARATE writing booklet.

(a) Find $\int \sin^3 \theta d\theta$. 2

(b) (i) Express $\frac{3x+1}{(x+1)(x^2+1)}$ in the form $\frac{a}{x+1} + \frac{bx+c}{x^2+1}$. 2

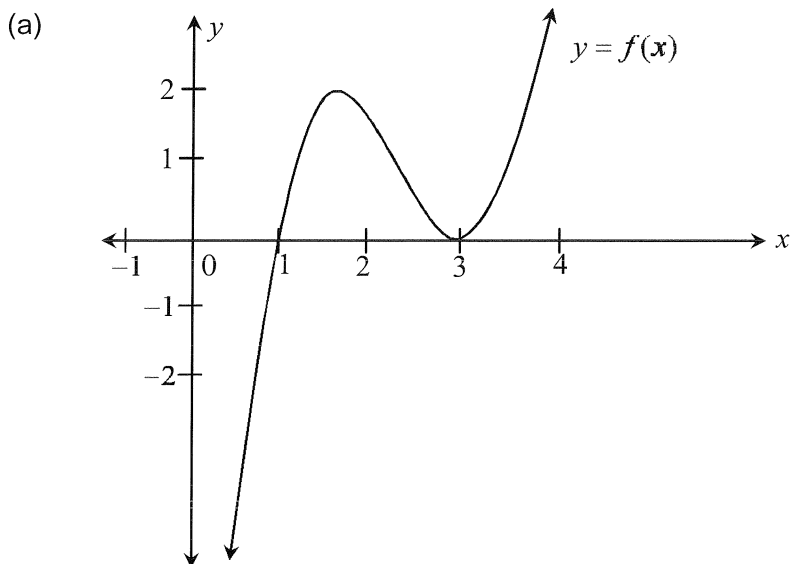
(ii) Hence find $\int \frac{3x+1}{(x+1)(x^2+1)} dx$. 2

(c) Use the substitution $x = 2\sin \theta$, or otherwise, to evaluate $\int_1^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}} dx$. 3

(d) Find $\int x^2 \sqrt{3-x} dx$. 3

(e) Evaluate $\int_0^1 \tan^{-1} \theta d\theta$. 3

QUESTION 2 (15 marks)
Start a new writing booklet.



The diagram above is a sketch of the function $y = f(x)$.

On separate diagrams sketch:

(i) $y = (f(x))^2$ 2

(ii) $y = \sqrt{f(x)}$ 2

(iii) $y = \ln f(x)$ 2

(iv) $y^2 = f(x)$ 2

(b) (i) If $f'(x) = \frac{2-x}{x^2}$ and $f(1) = 0$, find $f''(x)$ and $f(x)$. 3

(ii) Explain why the graph of $f(x)$ has only one turning point and find the value of the function at that point, stating whether it is a maximum or a minimum value. 2

(iii) Show that $f(4)$ and $f(5)$ have opposite signs and draw a sketch of $f(x)$. 2

QUESTION 3 (15 marks)
Start a new writing booklet.

(a) Express $(\sqrt{3} + i)^8$ in the form $x + iy$. 3

(b) On an Argand diagram, sketch the region where the inequalities 3

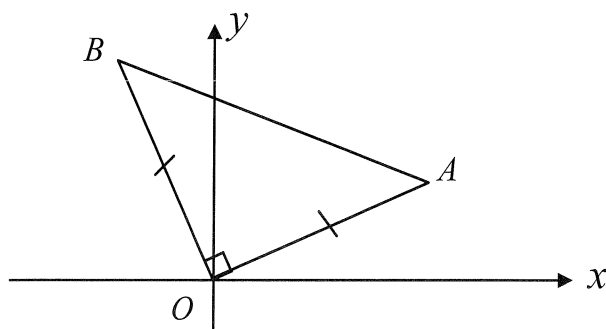
$$|z| \leq 3 \quad \text{and} \quad -\frac{2\pi}{3} \leq \arg(z+2) \leq \frac{\pi}{6} \quad \text{both hold.}$$

(c) Show that $\frac{1 + \sin \theta + i \cos \theta}{1 + \sin \theta - i \cos \theta} = \sin \theta + i \cos \theta$. 3

(d) (i) Express $z = \frac{-1+i}{\sqrt{3}+i}$ in modulus-argument form. 2

(ii) Hence evaluate $\cos \frac{7\pi}{12}$ in surd form. 2

(e) The Argand diagram below shows the points A and B which represent the complex numbers z_1 and z_2 respectively.



Given that $\triangle BOA$ is a right-angled isosceles triangle, show that $(z_1 + z_2)^2 = 2z_1z_2$. 2

QUESTION 4 (15 marks)**Start a new writing booklet.**

- (a) If $z = 1 + i$ is a root of the equation $z^3 + pz^2 + qz + 6 = 0$ where p and q are real, find p and q . 3

- (b) Show that if the polynomial $f(x) = x^3 + px + q$ has a multiple root, then $4p^3 + 27q^2 = 0$. 3

- (c) The base of a solid is the region in the first quadrant bounded by the curve $y = \sin x$, the x -axis and the line $x = \frac{\pi}{2}$. 3

Find the volume of the solid if every cross-section perpendicular to the base and the x -axis is a square.

- (d) (i) Find the five roots of the equation $z^5 = 1$. Give the roots in modulus-argument form. 2

- (ii) Show that $z^5 - 1$ can be factorised in the form :

$$z^5 - 1 = (z - 1)\left(z^2 - 2z \cos \frac{2\pi}{5} + 1\right)\left(z^2 - 2z \cos \frac{4\pi}{5} + 1\right) \quad 2$$

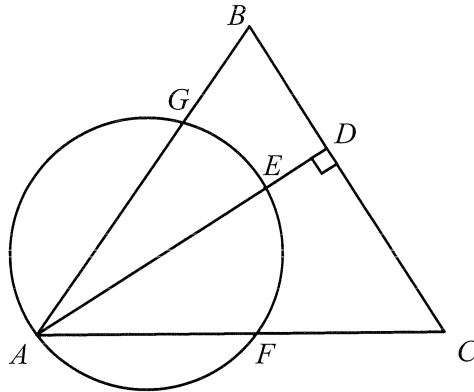
- (iii) Hence show that $\cos \frac{2\pi}{5} + \cos \frac{4\pi}{5} = -\frac{1}{2}$. 2

QUESTION 5 (15 marks)
Start a new writing booklet.

- (a) The ellipse $(x-1)^2 + \frac{y^2}{4} = 1$ is rotated about the y -axis.

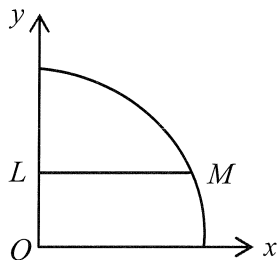
Use the method of slicing to find the volume of the solid formed by the rotation. 4

- (b) In the triangle ABC , AD is the perpendicular from A to BC . E is any point on AD and the circle drawn with AE as diameter cuts AC at F and AB at G . 4



Prove B, G, F and C are concyclic.

- (c) The diagram below shows the part of the circle $x^2 + y^2 = a^2$ in the first quadrant.



- (i) If the horizontal line LM through $L(0, b)$, where $0 < b < a$, divides the area between the curve and the coordinates axes into two equal parts, show that

$$\sin^{-1} \frac{b}{a} + \frac{b\sqrt{a^2 - b^2}}{a^2} = \frac{\pi}{4}.$$

3

- (ii) If the radius of the circle is 1 unit, show that b can be found by solving the equation

$$\sin 2\theta = \frac{\pi}{2} - 2\theta, \text{ where } \theta = \sin^{-1} b.$$

3

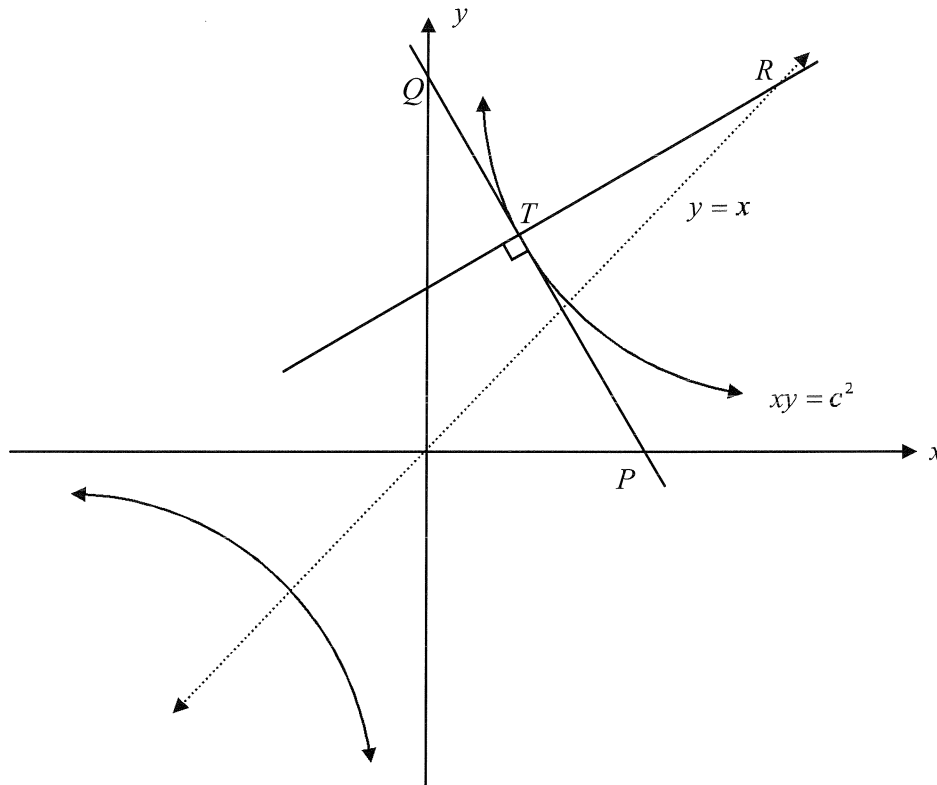
- (iii) Without attempting to solve the equation, how could θ (and hence b) be approximated? 1

QUESTION 6 (15 marks)
Start a new writing booklet.

- (a) An ellipse has equation $\frac{x^2}{4} + \frac{y^2}{3} = 1$ with vertices $A(2,0)$ and $A'(-2,0)$. P is a point (x_1, y_1) on the ellipse.
- (i) Find its eccentricity, coordinates of its foci, S and S' , and the equations of its directrices. **3**
 - (ii) Prove that the sum of the distances SP and $S'P$ is independent of the position of P . **3**
 - (iii) Show that the equation of the tangent to the ellipse at P is $\frac{xx_1}{4} + \frac{yy_1}{3} = 1$. **3**
 - (iv) The tangent at $P(x_1, y_1)$ meets the directrix at T . Prove that angle PST is a right angle. **3**
- (b) If $a + b + c = 1$,
- (i) Prove $a^2 + b^2 \geq 2ab$. **1**
 - (ii) Prove $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9$. **2**

QUESTION 7 (15 marks)
Start a new writing booklet.

- (a) The point $T(ct, \frac{c}{t})$ lies on the hyperbola $xy = c^2$.
 The tangent at T meets the x -axis at P and the y -axis at Q .
 The normal at T meets the line $y = x$ at R .



You may assume that the tangent at T has equation $x + t^2y = 2ct$.

- (i) Find the coordinates of P and Q . 2
- (ii) Find the equation of the normal at T . 2
- (iii) Show that the x -coordinate of R is $x = \frac{c}{t}(t^2 + 1)$. 2
- (iv) Prove that $\triangle PQR$ is isosceles. 3
- (b) (i) If $I_n = \int \frac{dx}{x^2 + 1}^n$ prove that $I_n = \frac{1}{2(n-1)} \left[\frac{x}{(x^2 + 1)^{n-1}} + (2n-3)I_{n-1} \right]$. 4
- (ii) Hence evaluate $\int_0^1 \frac{dx}{x^2 + 1}^2$. 2

QUESTION 8 (15 marks)
Start a new writing booklet.

Marks

- (a) A plane of mass M kg on landing, experiences a variable resistive force due to air resistance of magnitude Bv^2 newtons, where v is the speed of the plane. That is, $M\ddot{x} = -Bv^2$.

- (i) Show that the distance (D_1) travelled in slowing the plane from speed V to speed U under the effect of air resistance only, is given by: 4

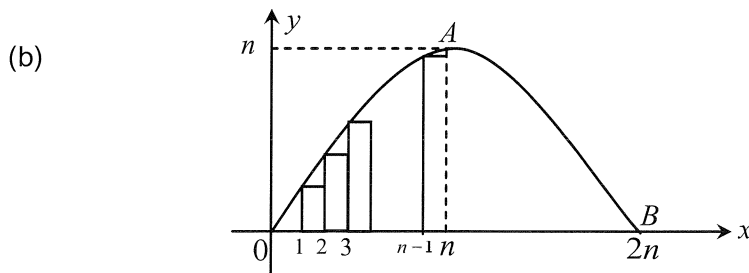
$$D_1 = \frac{M}{B} \ln\left(\frac{V}{U}\right)$$

After the brakes are applied, the plane experiences a constant resistive force of A Newtons (due to brakes) as well as a variable resistive force, Bv^2 . That is, $M\ddot{x} = -(A + Bv^2)$.

- (ii) After the brakes are applied when the plane is travelling at speed U , show that the distance D_2 required to come to rest is given by: 4

$$D_2 = \frac{M}{2B} \ln\left[1 + \frac{B}{A}U^2\right].$$

- (iii) Use the above information to estimate the total stopping distance after landing, for a 100 tonne plane if it slows from 90 m/s to 60 m/s under a resistive force of $125v^2$ Newtons and is finally brought to rest with the assistance of a constant braking force of magnitude 75 000 Newtons. (Note: 1 Newton (N) = 1 kg.m/s²) 2



The diagram above represents the curve $y = n \sin \frac{\pi x}{2n}$, $0 \leq x \leq 2n$, where n is any integer $n \geq 2$.

The points $O(0,0)$, $A(n,n)$ and $B(2n,0)$ lie on this curve.

- (i) By considering the areas of the lower rectangles of width 1 from $x=0$ to $x=n$, prove that

$$\sin \frac{\pi}{2n} + \sin \frac{2\pi}{2n} + \sin \frac{3\pi}{2n} + \dots + \sin \frac{\pi(n-1)}{2n} < \frac{2n}{\pi}. \quad 3$$

- (ii) Hence or otherwise, explain why $2n \sum_{r=1}^{n-1} \sin \frac{\pi r}{2n} < \frac{\pi n^2}{2}$. 2

END OF PAPER