Student Number:____________________

Student Name ____________________

Teacher Name: ____________________

ABBOTSLEIGH

AUGUST 2011
YEAR 12
ASSESSMENT 4

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.

Total marks – 120

- Attempt Questions 1-8.
- All questions are of equal value.
- Answer each question in a new booklet.

Outcomes assessed

HSC course

- **E1** appreciates the creativity, power and usefulness of mathematics to solve a broad range of problems
- **E2** chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings
- **E3** uses the relationship between algebraic and geometric representations of complex numbers and of conic sections
- **E4** uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials
- **E5** uses ideas and techniques from calculus to solve problems in mechanics involving resolution of forces and resisted motion
- **E6** combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions
- **E7** uses the techniques of slicing and cylindrical shells to determine volumes
- **E8** applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems
- **E9** communicates abstract ideas and relationships using appropriate notation and logical argument

Harder applications of the Extension 1 Mathematics course are included in this course. Thus the Outcomes from the Extension 1 Mathematics course are included.

From the Extension 1 Mathematics Course

Preliminary course

- **PE1** appreciates the role of mathematics in the solution of practical problems
- **PE2** uses multi-step deductive reasoning in a variety of contexts
- **PE3** solves problems involving inequalities, polynomials, circle geometry and parametric representations
- **PE4** uses the parametric representation together with differentiation to identify geometric properties of parabolas
- **PE5** determines derivatives that require the application of more than one rule of differentiation
- **PE6** makes comprehensive use of mathematical language, diagrams and notation for communicating in a wide variety of situations

HSC course

- **HE1** appreciates interrelationships between ideas drawn from different areas of mathematics
- **HE2** uses inductive reasoning in the construction of proofs
- **HE3** uses a variety of strategies to investigate mathematical models of situations involving projectiles, simple harmonic motion or exponential growth and decay
- **HE4** uses the relationship between functions, inverse functions and their derivatives
- **HE5** applies the chain rule to problems including those involving velocity and acceleration as functions of displacement
- **HE6** determines integrals by reduction to a standard form through a given substitution
- **HE7** evaluates mathematical solutions to problems and communicates them in an appropriate form

Total marks – 120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

QUESTION 1 (15 Marks) Use a SEPARATE writing booklet.

(a) Find $\int \frac{1+x}{4+x^2}\, dx$ 2

(b) By completing the square find $\int \frac{1}{\sqrt{6-x^2-x}}\, dx$ 2

(c) Find $\int \sin^3 x \cos^3 x\, dx$ 2

(d) Use integration by parts to evaluate $\int_0^{\ln 2} x e^{-x}\, dx$. Give your answer in simplest form. 3

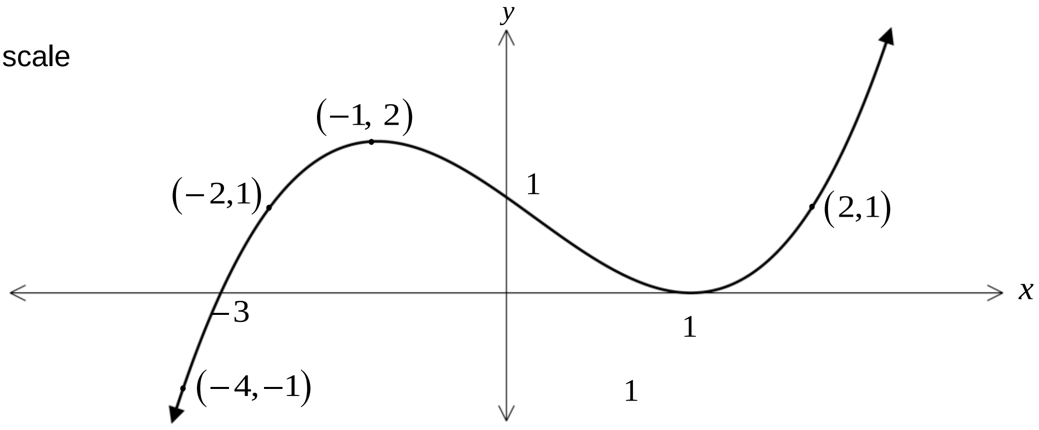
(e) By making the numerator rational, or otherwise, find $\int \sqrt{\frac{5-x}{5+x}}\, dx$ 3

(f) Use a trigonometric substitution to find $\int \frac{dx}{x^2\sqrt{4-x^2}}$ 3

Marks

QUESTION 2 (15 Marks) Use a SEPARATE writing booklet.

(a) The following is a sketch of a function $y = f(x)$



Draw separate one-third page sketches of the following: (clearly showing important features)

(i) $y = -f(x)$ 1

(ii) $y = \sqrt{f(x)}$ 1

(iii) $y = f(1-x)$ 2

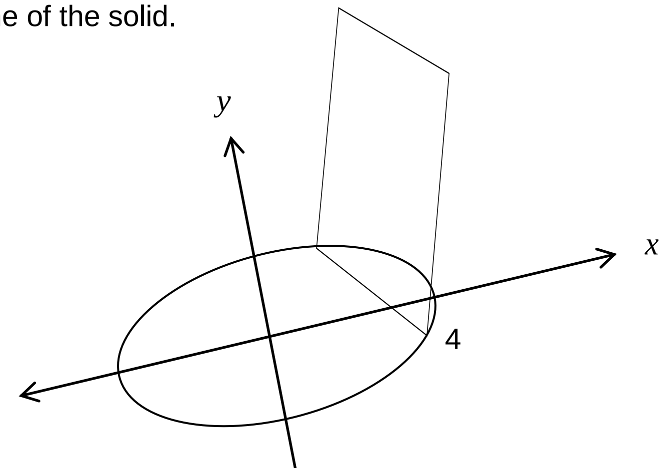
(iv) $y = \cos^{-1} f(x)$ 2

(v) $y = \dfrac{1}{1 - f(x)}$ 2

(b) Write down the equation of $P(x)$ if it is a monic polynomial of degree 3 with integer coefficients, a constant term of 12 and one root equal to $\sqrt{3}$. Leave your answer in factored form. 2

(c) Evaluate $\int_0^3 |x+1| \, dx$ 2

(d) The base of a solid is in the circle $x^2 + y^2 = 16$ and every plane section perpendicular to the x axis is a rectangle whose height is twice its base (which lies inside the circle). Find the volume of the solid. 3



Marks

QUESTION 3 (15 Marks) Use a SEPARATE writing booklet.

(a) Let $z = \dfrac{2-3i}{1+i}$

(i) Find $\overline{z}$ in the form $x + iy$ — 2

(ii) Evaluate $|z|$ — 1

(b) Consider $w = -\sqrt{3} + i$

(i) Express w in modulus-argument form — 2

(ii) Hence or otherwise show that $w^7 + 64w = 0$ — 2

(c) Sketch the region in the complex plane where the inequalities $1 \le |z - i| \le 2$ and $\mathrm{Im}(z) \ge 0$ hold simultaneously.
Clearly mark in all x *and* y intercepts. — 3

(d) In an Argand diagram z is a point on the circle $|z| = 2$.

Given that $\arg z = \theta$ and $0 < \theta < \dfrac{\pi}{2}$

(i) Draw a diagram to represent this information. — 1

(ii) Find, in terms of θ, an expression for $\arg z^2$ — 1

(iii) Find, in terms of θ, giving brief reasons, expressions for :

(A) $\arg(z+2)$ — 1

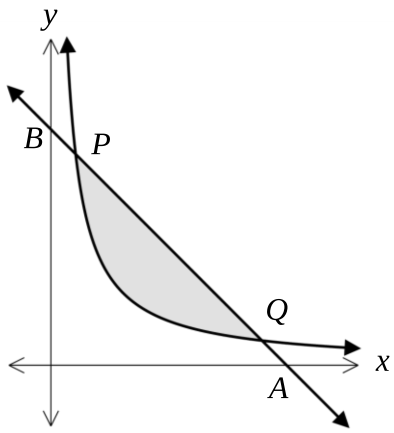
(B) $\arg(z-2)$ — 1

(C) $\left|\dfrac{z-2}{z+2}\right|$ — 1

Marks

QUESTION 4 (15 Marks) Use a SEPARATE writing booklet.

(a) Consider the rectangular hyperbola $xy = c^2$ where $c > 0$.



(i) Prove that the equation of the chord joining points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ where $0 < p < q$ is given by $x + pqy = c\left(p + q\right)$. 2

(ii) The chord PQ intersects the x and y axes at A and B respectively. Prove $AP = BQ$. 2

(iii) Show that the area enclosed by the hyperbola $xy = c^2$ and chord PQ is $\frac{c^2\left(q^2 - p^2\right)}{2pq} + c^2\ln\left(\frac{p}{q}\right)$ square units. 2

(b) (i) Divide the polynomial $P\left(x\right) = x^4 + 3x^3 - 7x^2 + 11x - 1$ by $x^2 + 2$ and write your result in the form $P\left(x\right) = \left(x^2 + 2\right)Q\left(x\right) + cx + d$. 2

(ii) Hence determine the values of a and b for which the polynomial $\left(x^4 + 3x^3 - 7x^2 + 2x\right) + ax + b$ is exactly divisible by $x^2 + 2$. 2

(c) The equation $\left|z - 3\right| + \left|z + 3\right| = 10$ corresponds to an ellipse in the Argand diagram.

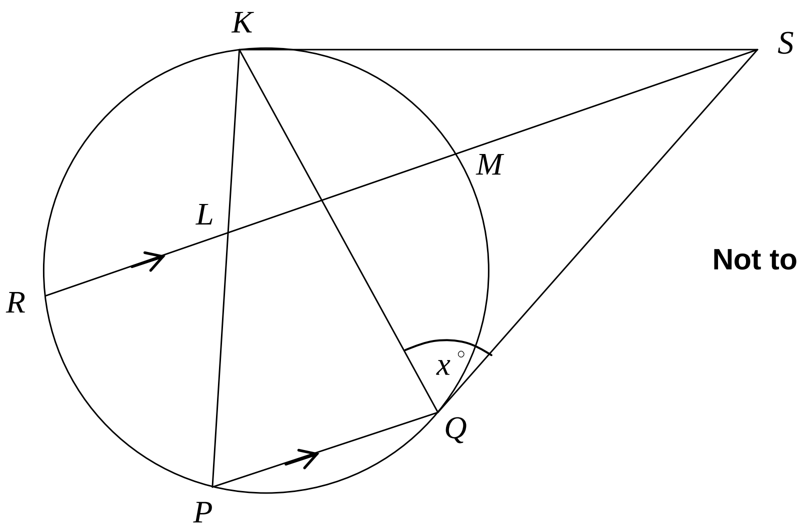
(i) Prove that the equation of the ellipse is $\frac{x^2}{25} + \frac{y^2}{16} = 1$ 3

(ii) Sketch the ellipse showing all important features. 2

QUESTION 5 (15 Marks) Use a SEPARATE writing booklet. **Marks**

(a) If $u_1 = 1$, $u_2 = 5$ and $u_n = 5u_{n-1} - 6u_{n-2}$ for integers $n \geq 3$, prove by induction that $u_n = 3^n - 2^n$ for integers $n \geq 1$. 3

(b) In the diagram below PQ and RM are parallel chords in a circle. The tangent at Q meets RM produced at S and SK is another tangent to the circle. PK cuts RM at L.



Not to Scale

(i) Copy or trace this diagram into your answer booklet.
Let $\angle SQK = x^\circ$ and prove $\angle SQK = \angle SLK$ 2

(ii) Explain why $LKSQ$ is a cyclic quadrilateral. 1

(iii) Prove $PL = QL$ 3

(c) It is given that $\sum_{r=0}^{n} (-1)^n \frac{{}^nC_r}{x+n} = \frac{n!}{x(x+1)(x+2)......(x+n)}$. (DO NOT PROVE)

Hence prove $1 - \frac{1}{2}\,{}^nC_1 + \frac{1}{3}\,{}^nC_2 - ....\frac{(-1)^n\,{}^nC_n}{n+1} = \frac{1}{n+1}$ 2

Question 5 continues on the next page.

(d) The curve $y = 8x - x^2$ and the line $y = 12$ is sketched below.
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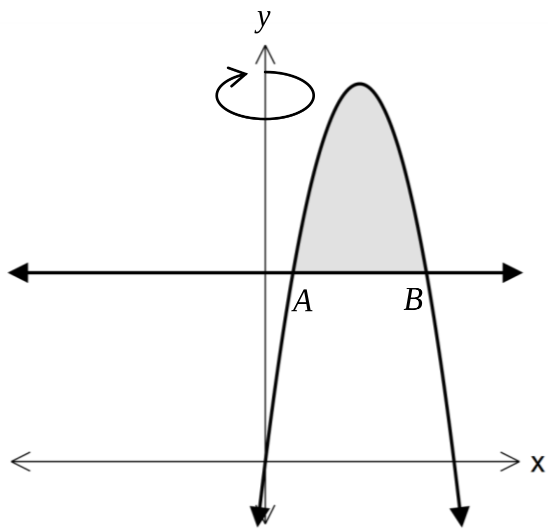

(i) Find the coordinates of the points of intersection A and B 1

(ii) The shaded area is rotated around the y axis.
Use the method of **cylindrical shells** to find the exact volume formed. (You may leave your answer **unsimplified** in fractional form) 3

Marks

QUESTION 6 (15 Marks) Use a SEPARATE writing booklet.

(a) (i) If $\dfrac{1}{x(\pi-2x)} = \dfrac{A}{x} + \dfrac{B}{\pi - 2x}$ and $A = \dfrac{1}{\pi}$ find B in terms of π. **1**

(ii) Hence show that $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{x(\pi - 2x)} = \frac{2}{\pi}\ln 2$ **3**

(iii) By using the substitution $u = a + b - x$ show that

$\displaystyle\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$ **1**

(iv) Hence evaluate $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^2 x\,dx}{x(\pi - 2x)}$ **3**

(b) A curve is defined by the equation $y = \dfrac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$

(i) Show that $1 + \left(\dfrac{dy}{dx}\right)^2 = \dfrac{y^2}{a^2}$ **3**

(ii) The arc length S between points $(0, a)$ and (x, y) of the curve is given by

$S = \displaystyle\int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$ (DO NOT PROVE THIS)

Show that $S = \sqrt{y^2 - a^2}$ **4**

Marks

QUESTION 7 (15 Marks) Use a SEPARATE writing booklet.

(a) (i) Explain why the domain of the function, $f(x) = \sqrt{2-\sqrt{x}}$ is $0 \le x \le 4$ **1**

(ii) Show that $f(x)$ is a decreasing function and hence find its range. **2**

(iii) Using the substitution, $u = 2-\sqrt{x}$ or otherwise, find the area bounded by the curve and the x and y axes. **3**

(b) Let $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$ where n is an integer and $n \ge 3$.

Show that $I_n + I_{n-2} = \frac{1}{n-1}$ **3**

(c) A body mass of 1 kg falls vertically downwards, from rest, in a medium which exerts a resistance to its motion of $\frac{1}{100}v^2$ Newtons (where v metres per second is the speed of the body when it has fallen a distance of x metres).

(i) Show (on a diagram) that the equation of motion of the body is $\ddot{x} = g - \frac{1}{100}v^2$ where g is the acceleration due to gravity. **1**

(ii) Show that the terminal speed V_T is given by $V_T = 10\sqrt{g}$ **2**

(iii) Prove that $v^2 = (V_T)^2\left(1 - e^{-\frac{x}{50}}\right)$ **3**

QUESTION 8 (15 Marks) Use a SEPARATE writing booklet. **Marks**

(a) A curve is defined implicitly by the equation $x^2 + 2xy + y^5 = 4$

(i) Show that the gradient of the tangent at $P\left(X, Y\right)$ is given by

$$\frac{dy}{dx} = \frac{-2X - 2Y}{5Y^4 + 2X}$$ 2

(ii) The tangent is horizontal at P. Show that X satisfies $X^5 + X^2 + 4 = 0$. 1

(iii) Show that X is the unique real solution of $X^5 + X^2 + 4 = 0$ and that $-2 < X < -1$ 3

(b) (i) Solve $\tan 4\theta = 1$ for $0 \le \theta \le \pi$ 1

(ii) Express $\tan 2\theta$ in terms of $\tan\theta$. 1

(iii) Hence show $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$. 2

(iv) Hence show $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$ has roots $\tan\dfrac{\pi}{16}$, $\tan\dfrac{5\pi}{16}$, $\tan\dfrac{9\pi}{16}$ and $\tan\dfrac{13\pi}{16}$. 2

(v) Hence evaluate $\tan\dfrac{\pi}{16}\tan\dfrac{5\pi}{16}\tan\dfrac{9\pi}{16}\tan\dfrac{13\pi}{16}$ 1

(vis1) By solving $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$ another way, show the exact value of $\tan\dfrac{\pi}{16} - \cot\dfrac{\pi}{16} = -2 - 2\sqrt{2}$. 2

END OF PAPER

TABLE OF STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \qquad x > 0$$

$$\int e^{ax} dx = \quad = \frac{1}{a} e^{ax}, \qquad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2})$$

$$NOTE: \quad \ln x = \log_e x, \qquad x > 0$$